

30 Marks

ALGEBRA OF MATRICES..

Multiplication of a matrix by a scalar of $A = [a_{ij}]$ and k is any scalar then

$$KA = k[a_{ij}]$$

$$= [ka_{ij}]$$

If $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ Then, $2A = \begin{bmatrix} 4 & 6 \\ 10 & 14 \end{bmatrix}$

Ques. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Find $2A - 3B$

Solution : $2A = \begin{bmatrix} 4 & 6 & 2 \\ 0 & -2 & 10 \end{bmatrix}$, $3B = \begin{bmatrix} 3 & 6 & -3 \\ 0 & -3 & 9 \end{bmatrix}$

$$2A - 3B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix}$$

CANCELLATION LAW :

$$A + B = A + C \Rightarrow B = C \quad \{ \text{Left Cancellation Law} \}$$

$$A + C = B + C \Rightarrow A = B \quad \{ \text{Right Cancellation Law} \}$$

Multiplication of two matrices :

$$A = [a_{ij}]_{m \times n} \text{ \& \, } B = [a_{ij}]_{n \times p}$$

$$AB = [a_{ij}]_{m \times p}$$

Ques If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 7 \\ 2 & 3 \\ 5 & 9 \end{bmatrix}$. Find AB and BA

Sol. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2 \times 2}$, $B = \begin{bmatrix} 1 & 7 \\ 2 & 3 \\ 5 & 9 \end{bmatrix}_{3 \times 2}$

$\Rightarrow AB$ doesn't exist.

$BA = \begin{bmatrix} 1 & 7 \\ 2 & 3 \\ 5 & 9 \end{bmatrix}_{3 \times 2} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2 \times 2}$

$BA = \begin{bmatrix} 1+21 & 2+28 \\ 2+9 & 4+12 \\ 5+27 & 10+36 \end{bmatrix} \Rightarrow \begin{bmatrix} 22 & 30 \\ 11 & 16 \\ 32 & 46 \end{bmatrix}$

Ques $\begin{bmatrix} 4 & 1 & 2 \\ 0 & 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 & 5 \\ -1 & 0 & -2 \\ 3 & 4 & 7 \end{bmatrix} =$

Ques Find the product of two matrices:

$\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}_{3 \times 3}$ and $\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}_{3 \times 3}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Null Matrix

2.5 Marks.

Ques If A, B, C are three matrices such that

$$A = [x \ y \ z]_{1 \times 3}, \quad B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}_{3 \times 3}$$

$$C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3 \times 1} \quad \text{Find } ABC.$$

$$AB = [x \ y \ z]_{1 \times 3} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}_{3 \times 3}$$

$$AB = [ax+hy+gz \quad hx+by+fz \quad gx+fy+cz]$$

$$ABC = [ax+hy+gz \quad hx+by+fz \quad gx+fy+cz] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\therefore ABC = [ax^2+hxy+gxz \quad hxy+by^2+fyz \quad gxz+fyx+cz^2]$$

Ques For 3 matrices A, B, C .

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Verify the relations, $A^2 = B^2 = C^2 = I$

$$AB = -BA$$

$$AC = -CA$$

$$BC = -CB$$

Soln

$$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 0+1 & 0 \\ 0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -(i^2) & 0 \\ 0 & -i^2 \end{bmatrix} \Rightarrow \begin{bmatrix} -(-1) & 0 \\ 0 & -(-1) \end{bmatrix}$$

$$\Rightarrow B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow C^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore A^2 = B^2 = C^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$= - \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$= -(AB)$$

$$\therefore BA = -AB$$

Imp

Q. let $f(x) = x^2 - 5x + 6$, find $f(A)$, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

$$f(x) = x^2 - 5x + 6$$

$$\therefore f(A) = A^2 - 5A + 6I.$$

$$A^2 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$5A = 5 \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix}$$

$$\Rightarrow f(A) = A^2 - 5A + 6I.$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5-10+6 & -1-0+0 & 2-5+6 \\ 9-10+0 & -2-5+6 & 5-15+6 \\ 0-5+0 & -1+5+0 & -2-0+6 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} 1 & -1 & 3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

Ques. PROPERTIES OF MATRIX MULTIPLICATION:

- (i) $A(BC) = (AB)C$ { Associative law? }
(ii) $A(B+C) = AB+AC$ { Distributive law? }

In general, $AB \neq BA$
 $(A+B)^2 \neq A^2 + B^2 + 2AB$

$$\begin{aligned}(A+B)^2 &= (A+B)(A+B) \\ &= AA + AB + BA + BB \\ &= A^2 + AB + BA + B^2 \\ \boxed{(A+B)^2} &= \boxed{A^2 + B^2 + AB + BA}\end{aligned}$$

COMMUTE

1. COMMUTE MATRIX

Two matrix A and B are said to be commute matrix if

$$\boxed{AB = BA}$$

2. ANTI-COMMUTE MATRIX

Two matrices A and B are said to be Anti-commute matrix if

$$\boxed{AB = -BA}$$

3. LOWER TRIANGULAR MATRIX

A square matrix $A = [a_{ij}]$ is said to be lower triangular matrix.

If $a_{ij} = 0$ where $i < j$

e.g.

$$\begin{bmatrix} a & 0 & 0 \\ e & b & 0 \\ d & f & c \end{bmatrix}$$

4. UPPER TRIANGULAR MATRIX:

A square matrix $A = [a_{ij}]$ is said to be upper triangular matrix if

$$a_{ij} = 0 \quad \text{when } i > j$$

e.g.

$$\begin{bmatrix} a & f & e \\ 0 & b & d \\ 0 & 0 & c \end{bmatrix}$$

Ques Prove that product of two matrix

$$\begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} \cdot \begin{bmatrix} \cos^2 \phi & \cos \phi \sin \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix}$$

is a zero matrix when θ & ϕ differ by an odd multiple of $\pi/2$.

$$= \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} \begin{bmatrix} \cos^2 \phi & \cos \phi \sin \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 \theta \cos^2 \phi + \cos \theta \sin \theta \cos \phi \sin \phi & \cos^2 \theta \cos \phi \sin \phi + \cos \theta \sin \theta \sin^2 \phi \\ \cos \theta \sin \theta \cos^2 \phi + \sin^2 \theta \cos \phi \sin \phi & \cos \theta \sin \theta \cos \phi \sin \phi + \sin^2 \theta \sin^2 \phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta \cos \phi (\cos \theta \cos \phi + \sin \theta \sin \phi) & \cos \theta \sin \phi (\cos \theta \cos \phi + \sin \theta \sin \phi) \\ \cos \phi \sin \theta (\sin \theta \cos \phi + \sin \theta \sin \phi) & \sin \theta \sin \phi (\cos \theta \cos \phi + \sin \theta \sin \phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta \cos \phi \cos(\theta - \phi) & \cos \theta \sin \phi \cos(\theta - \phi) \\ \cos \theta \sin \phi \cos(\theta - \phi) & \sin \theta \sin \phi \cos(\theta - \phi) \end{bmatrix}$$

Imp

Ques

If $A_\alpha = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$

show that $A_\alpha \cdot A_\beta = A_{\alpha+\beta} = A_\beta \cdot A_\alpha$

Solution

$$A_\beta = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix}$$

$$A_\alpha \cdot A_\beta = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos \alpha \cos \beta - \sin \alpha \sin \beta & -\cos \alpha \sin \beta - \sin \alpha \cos \beta \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta & -\sin \alpha \sin \beta + \cos \alpha \cos \beta \end{bmatrix}$$

$$A_\alpha \cdot A_\beta = \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix}$$

$$\Rightarrow A_\alpha \cdot A_\beta = A_{\alpha+\beta} \quad \text{--- (1)}$$

$$A_\beta \cdot A_\alpha = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha & -\cos\beta\sin\alpha - \sin\beta\cos\alpha \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha & -\sin\beta\sin\alpha + \cos\beta\cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$A_\beta \cdot A_\alpha = A_{\alpha+\beta} = A_\alpha \cdot A_\beta$$

Q. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ then $A^K = \begin{bmatrix} 1+2K & -4K \\ K & 1-2K \end{bmatrix}$

where K is any position integer.

Sol. Let $P(K) : A^K = \begin{bmatrix} 1+2K & -4K \\ K & 1-2K \end{bmatrix}$

For $K = 1$

$$P(1) : A^1 = \begin{bmatrix} 1+2(1) & -4(1) \\ 1 & 1-2(1) \end{bmatrix} \\ = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$\therefore P(K)$ is true for $K = 1$

Let for $K = n$ is true for $K = n$

$$P(n) : A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$$

For $K = n+1$

$$P(K+1) : A^{n+1} = \begin{bmatrix} 1+2(n+1) & -4(n+1) \\ n+1 & 1-2(n+1) \end{bmatrix}$$

$$P(K+1) : A^{n+1} = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+6n-4n & -4-8n+4n \\ 3n+1-2n & -4n-1+2n \end{bmatrix}$$

$$= \begin{bmatrix} 3+2n & -(n+1) \\ n+1 & -(2n+1) \end{bmatrix}$$

from Question

$$= \begin{bmatrix} 1+2+2n & -4(n+1) \\ (n+1) & 1-1-1-2n \end{bmatrix}$$

$$A^{n+1} = \begin{bmatrix} 1+2(n+1) & -4(n+1) \\ n+1 & 1-2(n+1) \end{bmatrix}$$

$P(k)$ is true for $k = n+1$

$\therefore P(k)$ is true for all $n \in \mathbb{N}$.

Ques If $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ then $A^n = \begin{bmatrix} a^n & b \frac{a^n - 1}{a - 1} \\ 0 & 1 \end{bmatrix}$

where $n \in \mathbb{N}$.

Sol. Let $P(k) : A^k = \begin{bmatrix} a^k & b \frac{a^k - 1}{a - 1} \\ 0 & 1 \end{bmatrix}$

for $k = 1$

$$P(1) : A^1 = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

$\therefore n = 1$, it is true for $n = 1$.

Let for $k = n$ is true, for $k = n$

$$P(k) : A^k = \begin{bmatrix} a^k & b \frac{a^k - 1}{a - 1} \\ 0 & 1 \end{bmatrix}$$

for $k = n+1$

$$\begin{aligned} A^{k+1} &= A^k \cdot A \\ &= \begin{bmatrix} a^k & b \frac{a^k - 1}{a - 1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} a^K a + 0 & a^K b + \frac{b(a^K - 1)}{a - 1} \\ 0 & 1 \end{bmatrix}$$

$$A^{n+1} = \begin{bmatrix} a^{K+1} & \frac{b(a^{K+1} - 1)}{a - 1} \\ 0 & 1 \end{bmatrix}$$

$P(K)$ is true for $K = n+1$

$\therefore P(K)$ is true $\forall n \in \mathbb{N}$

Q. Show that $A^n = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$

where $n \in \mathbb{N}$

Sol. Let $P(K) : A^K = \begin{bmatrix} \cos K\theta & -\sin K\theta \\ \sin K\theta & \cos K\theta \end{bmatrix}$

For $K = 1$

$$P(1) : A^1 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$n = 1$, it is true for $n = 1$

Let for $K = n$, is true for $K = n$

$$P(K) : A^K = \begin{bmatrix} \cos K\theta & -\sin K\theta \\ \sin K\theta & \cos K\theta \end{bmatrix}$$

For $K = n+1$

$$\begin{aligned} A^{K+1} &= A^K \cdot A \\ &= \begin{bmatrix} \cos K\theta & -\sin K\theta \\ \sin K\theta & \cos K\theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \end{aligned}$$

Q If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and

$(A+B)^2 = A^2 + B^2$, find a & b .

Sol.

$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$

$(A+B)^2 = A^2 + B^2$

$A^2 + B^2 + AB + BA = A^2 + B^2$

$AB + BA = 0$

$AB = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$
 $= \begin{bmatrix} a-b & 1+1 \\ 2a-b & 2+1 \end{bmatrix}$

$= \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix}$ — (i)

$BA = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$
 $= \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix}$

$= \begin{bmatrix} a+2 & -(a+1) \\ b-2 & -(b+1) \end{bmatrix}$ — (ii)

$\therefore AB + BA = 0$

From eq " (i) & (ii)

$$\Rightarrow \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix} + \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2a-b+2 & -a+1 \\ 2a-2 & -b+4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow 2a-2 = 0$$

$$2(a-1) = 0$$

$$\boxed{a = 1}$$

$$-b+4 = 0$$

$$\boxed{b = 4}$$

Ques Show that the matrix A and B are anti-commute
where $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$

Sol. $AB = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1-4 & 1+1 \\ 2-4 & 2+1 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & 2 \\ -2 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & -1-1 \\ 4-2 & -4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ 2 & -3 \end{bmatrix} = - \begin{bmatrix} -3 & 2 \\ -2 & 3 \end{bmatrix}$$

$$\therefore BA = -AB$$

$$AB = -BA$$

It is a Anti commute matrix

Ques If $AB = A$ and $BA = B$ show that $A^2 = A$ and $B^2 = B$

Sol. $AB = A$ — ① $BA = B$ — ②

Eq. ① multiplying both sides by A:

$$(AB)A = AA$$

$$A(BA) = A^2$$

$$AB = A^2$$

$$\boxed{A = A^2}$$

[Using eq. ②]

[Using eq. ①]

Eq. ② $BA = B$ multiplying both sides by B

$$(BA)B = BB$$

$$B(AB) = B^2$$

$$B \cdot A = B^2$$

$$\boxed{B = B^2}$$

[Using eq. ①]

[Using eq. ②]

5 INVOLUTRY MATRIX

Any square matrix A is said to be involutory matrix. If

$$A^2 = I$$

Ques Show that the matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutory.

Sol. $A^2 = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 = I$$

Given matrix A is an involutory matrix.

6. IDEMPOTENT MATRIX

Any square matrix A is said to be Idempotent matrix, if

$$\boxed{A^2 = A}$$

Ques Show that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

Solution $A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$\boxed{A^2 = A.}$$

Given matrix A is an Idempotent matrix.

NILPOTENT MATRIX :

Any square matrix A is said to be Nilpotent matrix of order m . If

$$\boxed{A^m = 0}$$

When $A^{m-1} \neq 0$

Q. Show that the matrix $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ is nilpotent matrix.

$$1. \quad A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

$$= \begin{bmatrix} a^2b^2 - a^2b^2 & ab \cdot b^2 - ab \cdot b^2 \\ -a^2 \cdot ab - ab \cdot a^2 & -a^2b^2 - ab \cdot ab \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Given matrix is a nilpotent matrix where order 2.

Q. Show that the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is a nilpotent matrix of order = 3.

$$4. \quad A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Given matrix A is a Nilpotent matrix where
Order = 3.

• TRANSPOSE OF A MATRIX

The matrix of order $m \times n$ obtained by interchanging the rows and columns of a matrix of order $n \times m$ is called the transpose of a matrix.

And it is denoted by A' .

Example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 7 & -3 \end{bmatrix}_{3 \times 2}$

Transpose of a matrix.

$$A' = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 4 & -3 \end{bmatrix}$$

PROPERTIES OF A TRANSPOSE OF A MATRIX:

- (i) $(A')' = A$
- (ii) $(kA)' = kA'$
- (iii) $(A+B)' = A' + B'$
- (iv) $(AB)' = B'A'$

Q If $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ Then using that $(AB)' = B'A'$

$$A' = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}, \quad B' = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6+6 & 8+3 \\ 2 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 12 & 11 \\ 2 & 1 \end{bmatrix}$$

$$(AB)' = \begin{bmatrix} 12 & 2 \\ 11 & 1 \end{bmatrix} \quad \text{--- (i)}$$

$$\Rightarrow B'A' = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6+6 & 2 \\ 8+3 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 2 \\ 11 & 1 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 12 & 2 \\ 11 & 1 \end{bmatrix} \quad \text{--- (ii)}$$

From eqⁿ (i) & (ii)

$$(AB)' = B'A' \quad \text{Proved}$$

8. ^{Syllabus} **SYMMETRIC MATRIX**: Any square matrix $A = [a_{ij}]$

is said to be symmetric matrix if

$$A' = A \quad \text{or} \quad a_{ij} = a_{ji} \quad \forall i, j$$

Example $\rightarrow \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix.

9. ^{Syllabus} **SKEW SYMMETRIC MATRIX**:

A square matrix $A = [a_{ij}]$ is said to be skew symmetric matrix if

$$A' = -A \quad \text{or} \quad a_{ij} = -a_{ji} \quad \forall i, j$$

Example $\rightarrow \begin{bmatrix} 0 & 2 & -7 \\ -2 & 0 & -8 \\ 7 & 8 & 0 \end{bmatrix}$ is a skew symmetric matrix.

Imp

Ques Prove that every diagonal elements of a skew symmetric matrix are equal to zero.

Proof : In a skew symmetric matrix.

$$A' = -A$$

$$\text{or } a_{ij} = -a_{ji} \quad \forall i, j$$

For main diagonal elements $i=j$
 $a_{ij} = -a_{ij}$ ($\because j=i$)

$$a_{ii} = -a_{ii}$$

$$a_{ii} + a_{ii} = 0$$

$$\Rightarrow 2(a_{ii}) = 0$$
$$\Rightarrow a_{ii} = 0$$

$$\therefore a_{11} = a_{22} = a_{33} = 0$$

$\therefore$ Every diagonal element in a skew symmetric matrix are equal to zero.

★ 2013 Most Imp Theorem.

Prove that every square matrix can be uniquely expressed as the sum of a symmetric and skew symmetric matrix.

Sol. let A be a square matrix.

$$A = \frac{1}{2} (A + A')$$

$$= \frac{1}{2} [(A + A') + (A - A')]$$

$$= \frac{1}{2} (A + A') + \frac{1}{2} (A - A')$$

$$A = \underset{\downarrow}{P} + \underset{\downarrow}{Q} \quad \text{--- (1)}$$

$$A = P + Q \quad \text{--- (1)}$$

$$P' = \left[\frac{1}{2} (A+A') \right]'$$

By the properties of transpose of matrix

$$(KA)' = KA'$$

$$(A+B)' = A' + B'$$

$$(A')' = A$$

$$= \frac{1}{2} (A+A')'$$

$$= \frac{1}{2} [A' + (A')']$$

$$= \frac{1}{2} [A' + A]$$

$$= \frac{1}{2} [A + A']$$

$$\boxed{P' = P}$$

$\therefore P = \frac{1}{2} (P+P')$ is a symmetric matrix.

$$Q' = \left[\frac{1}{2} (A-A') \right]'$$

$$= \frac{1}{2} (A-A')'$$

$$= \frac{1}{2} [A' - (A')']$$

$$= \frac{1}{2} [A' - A]$$

$$= -\frac{1}{2} (A-A')$$

$$\boxed{Q' = -Q}$$

$\therefore Q = \frac{1}{2} (A-A')$ is skew symmetric matrix.

$$\therefore A = \frac{1}{2} (A+A') + \frac{1}{2} (A-A')$$

$\downarrow$ Symmetric matrix $\searrow$ skew symmetric matrix.

$\therefore$ Every square matrix can be expressed as the sum of symmetric and skew symmetric matrix.

★ UNIQUENESS : let $A = A_1 + A_2$ — (i)

$\downarrow$ Symmetric $\downarrow$ skew symmetric.

$A' = A_1$, $A_2' = -A_2$

$$A' = (A_1 + A_2)'$$

$$A' = A_1' + A_2'$$

$$A' = A_1 - A_2$$
 — (ii)

Adding eq. (i) & (ii)

$$A + A' = 2A_1$$

$$A_1 = \frac{1}{2} (A + A') = P.$$

Sub. eq. (ii) & (iii)

$$A = A' = 2A_2$$

$$A_2 = \frac{1}{2} (A - A') = Q$$

$\therefore A = P + Q$ { which is unique ? }

$= A_1 + A_2$

$\therefore$ Every square matrix can be uniquely expressed as the sum of a symmetric & skew symmetric matrix.

CONJUGATE OF A MATRIX.

The matrix obtain for any given matrix A on replacing its elements by the corresponding conjugate complex number is called the conjugate of A .

& it's denoted by $\bar{A}$.

Example : $A = \begin{bmatrix} 2+3i & -5i \\ 3-7i & 2+i \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 2-3i & 5i \\ 3+7i & 2-i \end{bmatrix}$$

Syllabus

10. HERMITIAN MATRIX

→ 2.5 marks

Any square matrix $A = [a_{ij}]$ is said to be hermitian matrix, if

$$\bar{a}_{ij} = a_{ji} \quad \forall i, j$$

Or

$$\bar{A}' = A$$

Example → $\begin{bmatrix} a & a+ib & x-iy \\ a-ib & b & 3i \\ x+iy & -3i & 0 \end{bmatrix}$

is a Hermitian matrix.

Imp
 * Every diagonal element of a Hermitian matrix are real.

$$\bar{A}^T = A$$

In a Hermitian matrix $\bar{a}_{ij} = a_{ji}$ — (1)

For main diagonal element $i = j$

$$\bar{a}_{ii} = a_{ii}$$

$$\overline{x+iy} = (x+iy)$$

$$x-iy = x+iy$$

$$-2iy = 0$$

$$y = 0$$

$$a_{ii} = x + i(0)$$

$$a_{ii} = x \quad (\text{purely real})$$

$$a_{11}, \bar{a}_{22}, \bar{a}_{33}, \dots \text{purely real.}$$

syllabus

II. SKEW HERMITIAN MATRIX :

Any square matrix $A = [a_{ij}]$ is said to be skew hermitian matrix if

$$\bar{A}' = -A \quad \text{Or}$$

$$\bar{a}_{ij} = -a_{ji} \quad \forall i, j$$

eg. $\rightarrow \begin{bmatrix} 2i & 3-2i & 1+i \\ -3-2i & -3i & -3+4i \\ -1+i & 3+4i & 0 \end{bmatrix}$ is a skew hermitian matrix.

* Every diagonal element of a skew hermitian matrix are purely imaginary or zero.

In a skew hermitian matrix $\bar{a}_{ij} = -a_{ji}$ — (1)

For main diagonal element, $i = j$

$$\bar{a}_{ii} = -a_{ii}$$

$$\overline{(x+iy)} = -(x+iy)$$

$$x-iy = -x-iy$$

$$2x = 0$$

$$\boxed{x = 0}$$

$$a_{ii} = x+iy = 0 + iy = iy \quad (\text{purely imaginary})$$

$$a_{11} = a_{22} = a_{33} = \dots = iy \quad (\text{purely imaginary})$$

Imp →

Q. Show that $A = \begin{bmatrix} i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix}$, Prove it

a skew hermitian matrix.

Sol $\bar{A} = \begin{bmatrix} i & -3+2i & -2-i \\ 3+2i & 0 & -3-4i \\ -2-i & 3-4i & -2i \end{bmatrix}$

$$\bar{A}' = \begin{bmatrix} -i & -3-2i & 2+i \\ 3-2i & 0 & -3+4i \\ -2+i & 3+4i & 2i \end{bmatrix}$$

$$\bar{A}' = - \begin{bmatrix} i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix}$$

$$\boxed{\bar{A}' = -A}$$

$$\Rightarrow \bar{A}' = A^\ominus = -A$$

Ques. If A be any symmetric matrix, then prove that AA' and $A'A$ are both symmetric matrices.

Sol. $\because A$ is a symmetric matrix.

$$\therefore A' = A$$

$$\begin{aligned} (AA')' &= (A')'(A) \\ &= AA' \end{aligned}$$

$$\left\{ \begin{aligned} (AB)' &= B'A' \\ (A')' &= A \end{aligned} \right\}$$

$$(AA')' = AA'$$

$\therefore AA'$ is a symmetric matrix

$$(A'A)' = (A')(A')'$$

$$= A'A$$

$$(A'A)' = A'A$$

$$\begin{cases} (AB)' = B'A' \\ (A')' = A \end{cases}$$

$\therefore A'A$ is a symmetric matrix.

Ques If A be any square ^{symm.} matrix then show that $A+A'$ is symmetric and $A-A'$ is a skew symmetric matrix.

Sol: $\because (A+A')$ is a symmetric matrix.

$$\because A' = A$$

$$(A+A')' = (A')' + (A)'$$

$$= A' + A$$

$$\boxed{(A+A')' = A' + A}$$

$$\begin{cases} \because (A+B)' = A' + B' \\ (A')' = A \end{cases}$$

$A+A'$ is a symmetric matrix.

$\because (A-A')$ is a skew symmetric matrix.

$$(A-A')' = A' - (A')'$$

$$= A' - A$$

$$\begin{cases} \because (A')' = A \end{cases}$$

$$(A-A')' = -(A-A')$$

$\therefore (A-A')$ is a skew symmetric matrix.

H.W If A and B are symmetric matrices of the same order then prove that $AB+BA$ is symmetric and $AB-BA$ is skew symmetric.

Sol $\rightarrow$

$\therefore A$ and B is a symmetric matrix.

$$\therefore A' = A, \quad B' = B.$$

$$\begin{aligned}(AB+BA)' &= (AB)' + (BA)' \\ &= B'A' + A'B'\end{aligned} \quad \left\{ \begin{aligned}(A+B)' &= A'+B' \\ (AB)' &= B'A'\end{aligned} \right\}$$

$$(AB+BA)' = BA + AB.$$

$$(AB+BA)' = AB + BA$$

$\rightarrow AB+BA$ is a symmetric matrix.

$$\begin{aligned}(AB-BA)' &= (AB)' - (BA)' \\ &= B'A' - A'B'\end{aligned} \quad \left\{ \begin{aligned}(A-B)' &= A'-B' \\ (AB)' &= B'A'\end{aligned} \right\}$$

$$(AB-BA)' = BA - AB.$$

$$(AB-BA)' = -(AB-BA).$$

$\rightarrow \therefore AB-BA$ is a skew symmetric matrix.

ADJOINT OF A MATRIX..

If A_{ij} (C_{ij}) be the cofactors of the element

a_{ij} . Then adjoint $A = [A_{ij}]'$

$$= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}'$$

$$= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Find the adjoint of the matrix : $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 2 & 0 & 4 \end{bmatrix}$

Ans $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 2 & 0 & 4 \end{bmatrix}$

$A_{11} = 4$, $A_{12} = -2$, $A_{13} = -2$

$A_{21} = -4$, $A_{22} = -2$, $A_{23} = +2$

$A_{31} = -4$, $A_{32} = 1$, $A_{33} = 1$

$= \begin{bmatrix} 4 & -2 & -2 \\ -4 & -2 & 2 \\ -4 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & -4 & -4 \\ -2 & -2 & 1 \\ -2 & 2 & 1 \end{bmatrix}$

12. INVERTIBLE MATRIX

If $|A| \neq 0$

Then, matrix A is said to be invertible matrix.

If $|A| \neq 0$, then matrix A is said to be non-singular matrix.

If $|A| = 0$ Then, matrix A is said to be singular matrix.

Q. Find n if the matrix $A = \begin{bmatrix} n & 2 \\ 3 & 1 \end{bmatrix}$ are singular

Sol. $|A| = \begin{vmatrix} n & 2 \\ 3 & 1 \end{vmatrix}$

$$|A| = n - 6$$

$\therefore$ for singular matrix A , $|A| = 0$

$$\therefore n - 6 = 0$$

$$n = 6$$

INVERSE OF THE MATRIX

If $|A| \neq 0$.

Then A^{-1} exists.

Then, $A^{-1} = \frac{1}{|A|} \text{adj} A$

Q. If $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ show that $A^{-1} = A$.

$$|A| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 0(0) - 0(0) + 1(0 - 1)$$

$$= -1$$

$$\therefore |A| \neq 0 = -1$$

$\therefore A^{-1}$ exists.

$$A_{11} = 0, A_{12} = 0, A_{13} = -1$$

$$A_{21} = 0, A_{22} = -1, A_{23} = 0$$

$$A_{31} = -1, A_{32} = 0, A_{33} = 0$$

$$\text{adj } A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\therefore A^{-1} = \frac{1}{-1} \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \boxed{A^{-1} = A}$$

Ques Find the inverse of the matrix

$$A = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$$

$$\begin{aligned} |A| &= (a+ib)(a-ib) - (c+id)(-c+id) \\ &= a^2 - i^2 b^2 - (i^2 d^2 - c^2) \\ &= a^2 - (-1)b^2 - (-1)d^2 + c^2 \\ &= a^2 + b^2 + d^2 + c^2 \\ &= a^2 + b^2 + c^2 + d^2 \neq 0 \end{aligned}$$

$$\therefore |A| \neq 0$$

$\therefore A^{-1}$ exists

$$A_{11} = a-ib$$

$$A_{12} = c-id$$

$$A_{21} = -c-id$$

$$A_{22} = a+ib$$

$$\text{adj} A = \begin{bmatrix} a-ib & -(c+id) \\ c-id & a+ib \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj} A$$

$$\therefore A^{-1} = \frac{1}{a^2+b^2+c^2+d^2} \begin{bmatrix} a-ib & -(c+id) \\ c-id & a+ib \end{bmatrix}$$

Ques find the inverse of the matrix.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\text{Sol } |A| = 2(3-0) - 0 + (-1)(5-0) \\ = 6 - 5 = 1$$

$$|A| = 1 \neq 0$$

$\therefore A^{-1}$ exists

factor

$$A_{11} = 3, A_{12} = -15, A_{13} = 5$$

$$A_{21} = +1, A_{22} = 6, A_{23} = -2$$

$$A_{31} = 0, A_{32} = +5, A_{33} = 2$$

$$\text{adj} A = \begin{bmatrix} 3 & 1 & 0 \\ -15 & 6 & 5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj} A$$

$$\therefore A^{-1} = \frac{1}{1} \begin{bmatrix} 3 & 1 & 0 \\ -15 & 6 & 5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 1 & 0 \\ -15 & 6 & 5 \\ 5 & -2 & 2 \end{bmatrix}$$

Ques. Find the inverse of the matrix.

$$A = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol. $|A| = \cos \theta \cdot \cos \theta + \sin \theta \sin \theta + 0$
 $= \cos^2 \theta + \sin^2 \theta$

$$|A| = 1 \neq 0$$

$$\therefore |A| \neq 0$$

$\therefore A^{-1}$ exists:

$$A_{11} = \cos \theta, A_{12} = -\sin \theta, A_{13} = 0$$

$$A_{21} = \sin \theta, A_{22} = \cos \theta, A_{23} = 0$$

$$A_{31} = 0, A_{32} = 0, A_{33} = 1.$$

$$\text{adj. } A = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} [\text{adj. } A]$$

$$\therefore A^{-1} = \frac{1}{1} \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: If A & B are two non-singular matrices. Then,
 $|AB| = |A||B|$

Theorem If A is non-singular matrix. Then,
 $|adj A| = |A|^{n-1}$

Proof →

$$A \cdot (adj A) = (adj A) \cdot A = |A| I \quad \text{--- (1)}$$

$$A \cdot (adj A) = |A| I$$

$$A \cdot (adj A) = |A| \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 & 0 & 0 \\ 0 & |A| & 0 & 0 & 0 \\ 0 & 0 & |A| & 0 & 0 \\ 0 & 0 & 0 & |A| & 0 \\ 0 & 0 & 0 & 0 & |A| \end{bmatrix}$$

$$|A \cdot (adj A)| = \begin{vmatrix} |A| & 0 & 0 & 0 & 0 \\ 0 & |A| & 0 & 0 & 0 \\ 0 & 0 & |A| & 0 & 0 \\ 0 & 0 & 0 & |A| & 0 \\ 0 & 0 & 0 & 0 & |A| \end{vmatrix}$$

$$|A| |adj A| = \begin{vmatrix} |A| & 0 & 0 & 0 & 0 \\ 0 & |A| & 0 & 0 & 0 \\ 0 & 0 & |A| & 0 & 0 \\ 0 & 0 & 0 & |A| & 0 \\ 0 & 0 & 0 & 0 & |A| \end{vmatrix}$$

$$|A| |adj A| = |A|^n (1)$$

$$\therefore |adj A| = \frac{|A|^n}{|A|}$$

$$|adj A| = |A|^{n-1} \quad \text{--- (1)}$$

Imp

Theorem: If A is a non-singular matrix of order $n \times n$.
Then, $|\text{adj}(\text{adj} A)| = |A|^{(n-1)^2}$

Proof: $\because A$ is a non-singular matrix.
 $\therefore |A| \neq 0$.

$$A \cdot (\text{adj} A) = (\text{adj} A) \cdot A = |A| I$$

$$A \cdot (\text{adj} A) = |A| \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & - \\ 0 & 1 & 0 & 0 & 0 & - \\ 0 & 0 & 1 & 0 & 0 & - \\ - & - & - & 1 & 0 & - \\ - & - & - & 0 & 1 & - \\ - & - & - & - & - & 1 \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 & 0 & 0 & - \\ 0 & |A| & 0 & 0 & 0 & - \\ 0 & 0 & |A| & 0 & 0 & - \\ 0 & 0 & 0 & |A| & 0 & - \\ - & - & - & - & - & 1 \end{bmatrix}$$

$$|A \cdot (\text{adj} A)| = \begin{vmatrix} |A| & 0 & 0 & 0 & 0 & - \\ 0 & |A| & 0 & 0 & 0 & - \\ 0 & - & |A| & 0 & 0 & - \\ - & - & - & |A| & 0 & - \\ - & - & - & - & |A| & - \\ - & - & - & - & - & 1 \end{vmatrix}$$

$$|A| \cdot |\text{adj} A| = |A|^n (1)$$

$$|A| \cdot |\text{adj} A| = |A|^n$$

$$|\text{adj} A| = |A|^{n-1} \quad \text{--- (1)}$$

Replacing A by $(\text{adj} A)$

$$|\text{adj}(\text{adj} A)| = |\text{adj} A|^{n-1}$$

$$= |A|^{n-1} \quad \{ \text{From eq (1)} \}$$

$$\Rightarrow |\text{adj}(\text{adj} A)| = |A|^{(n-1)^2}$$

Not Imp $\rightarrow$ (2012)

REVERSAL LAW FOR THE INVERSE OF A PRODUCT:

If A & B be two n -rowed non-singular matrices.
Then, AB is also non-singular and
 $(AB)^{-1} = B^{-1}A^{-1}$

Proof $\rightarrow$ $\because$ A and B are two non-singular square matrices
 $\therefore |A| \neq 0, |B| \neq 0$

$$\because |AB| = |A| |B| \neq 0 \quad \left\{ \because |A| \neq 0 \text{ \& } |B| \neq 0 \right\}$$
$$\therefore |AB| \neq 0$$

Hence, AB is non-singular matrix.

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$
$$= AIA^{-1}$$

$$\text{Note: } AB = BA = I$$

$$\Rightarrow A^{-1} = B$$

$$= AA^{-1}$$

$\left\{ \because B \text{ is a non-singular matrix} \right\}$

$$(AB)(B^{-1}A^{-1}) = I \quad \text{--- (i)}$$

$$\therefore BB^{-1} = I, A^{-1}A = I$$

$$\{ AA^{-1} = I \}$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$$

$$= B^{-1}IB$$

$$\left\{ \because A^{-1}A = I \right\}$$

$$= B^{-1}B$$

$$\left\{ \because B^{-1}I = B^{-1} \right\}$$

$$= I$$

$$\text{--- (ii)}$$

From eq (i) & (ii)

$$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$$

$$\Rightarrow$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Syllabus

13. ORTHOGONAL MATRIX..

A square matrix A is said to be orthogonal if $AA' = I$.

Q $\Rightarrow$ Verify that the matrix..

is orthogonal.

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$$

Sol. : $A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$

$$A' = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$A \cdot A' = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} + \frac{1}{6} + \frac{1}{2} & \frac{1}{3} - \frac{2}{6} + 0 & \frac{1}{3} + \frac{1}{6} - \frac{1}{2} \\ \frac{1}{3} - \frac{2}{6} + 0 & \frac{1}{3} + \frac{4}{6} + 0 & \frac{1}{3} - \frac{2}{6} + 0 \\ \frac{1}{3} + \frac{1}{6} - \frac{1}{2} & \frac{1}{3} - \frac{2}{6} + 0 & \frac{1}{3} + \frac{1}{6} + \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2+1+3}{6} & \frac{2-2}{6} & \frac{2+1-3}{6} \\ 0 & \frac{2+1}{6} & 0 \\ \frac{2+1-3}{6} & 0 & \frac{2+1+3}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow I$$

$$\Rightarrow A A' = I.$$

$\therefore$ Given matrix is orthogonal.

Q. Show that the matrix $\frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is orthogonal.

Sol $A = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$

$$A' = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$$

$$A \cdot A' = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{9} + \frac{4}{9} + \frac{4}{9} & -\frac{2}{9} - \frac{2}{9} + \frac{4}{9} & -\frac{2}{9} + \frac{4}{9} - \frac{2}{9} \\ -\frac{2}{9} - \frac{2}{9} + \frac{4}{9} & \frac{4}{9} + \frac{1}{9} + \frac{4}{9} & \frac{4}{9} + \frac{2}{9} - \frac{1}{9} \\ -\frac{2}{9} + \frac{4}{9} - \frac{2}{9} & \frac{4}{9} + \frac{2}{9} - \frac{2}{9} & \frac{4}{9} + \frac{4}{9} + \frac{1}{9} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

14. UNITARY MATRIX

Syllabus

A square matrix A is said to be unitary

if $A^H A = I$.

$$A^H = A^* = \bar{A}^T$$

conjugate transpose of A

Q. Prove that the matrix.

$$\begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1-i}{2} & \frac{1-i}{2} \end{bmatrix} \text{ is unitary}$$

Sol: $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1-i}{2} & \frac{1-i}{2} \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} \frac{1-i}{2} & \frac{-1-i}{2} \\ \frac{1+i}{2} & \frac{1+i}{2} \end{bmatrix}$$

$$\text{or } A^H = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$

$$A^{\Theta} A = \begin{bmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1-i}{2} & \frac{1-i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-i^2 + 1-i^2}{4} & \frac{-(1-i)^2 + (1-i)^2}{4} \\ \frac{-(1+i)^2 + (1+i)^2}{4} & \frac{+(1-i)^2 + 1-i^2}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 4/4 & 0 \\ 0 & 4/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore A^{\Theta} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^{\Theta} A = I$$

$\therefore$ Given matrix A is a unitary matrix.

Q Prove that $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary.

$$\text{Sol/ } A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & (1+i)/\sqrt{3} \\ (1-i)/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$$

$$\text{N/A} \Rightarrow \bar{A} = \begin{bmatrix} 1/\sqrt{3} & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} 1/\sqrt{3} & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & -1/\sqrt{3} \end{bmatrix}$$

$$A^{\theta} \cdot A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} + \frac{(1-i)^2}{3} & \frac{1-i - (1+i)}{3} \\ \frac{1-i - (1-i)}{3} & \frac{1-i^2 + 1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{3} & 0 \\ 0 & \frac{3}{3} \end{bmatrix}$$

$$A^{\theta} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^{\theta} A = I.$$

Given matrix is a unitary matrix.

Theorem: The inverse of a matrix is unique.

Proof: let B and C are the two inverse of A.

$$AB = BA = I \quad \text{--- (I)}$$

$$AC = CA = I \quad \text{--- (II)}$$

from eq. (I) $AB = I$.

$$C(AB) = CI$$

$$(CA)B = CI$$

$$IB = CI$$

$$\boxed{B = C}$$

{ using associative law,
 $A(BC) = (AB)C$.

{ "From eq" (II) $CA = I$.

Hence, inverse of A is unique.

Syllabus

ELEMENTARY TRANSFORMATION

Six transformation

3 Row Transformation

$$(i) R_i \leftrightarrow R_j$$

$$(ii) R_i \rightarrow kR_i$$

$$(iii) R_i \rightarrow R_i + kR_j$$

For $n \times n$

3 Column Transformation

$\leftrightarrow$ interchange

(x) Not in syllabus

For 3×3 matrix,

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$A = IA$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Q Find the inverse of the matrix

2.5 Marks

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Sol

$$A = IA$$

$$\therefore \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_2 \rightarrow R_2 + R_1$$

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} A$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & -2 & 1 \\ 0 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} A$$

$$R_3 \rightarrow R_3 + 2R_2$$

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} A$$

$$R_3 \rightarrow 2R_3 + R_2$$

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & 5 \end{bmatrix} A$$

$$R_1 \rightarrow R_1 + 2R_3, \quad R_2 \rightarrow R_2 - R_3$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 10 \\ -2 & -2 & -4 \\ 2 & 2 & 5 \end{bmatrix} A$$

$$R_1 \rightarrow R_1 + R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ -2 & -2 & -4 \\ 2 & 2 & 5 \end{bmatrix} A$$

$$R_2 \rightarrow \frac{1}{(-2)} R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A$$

$$I = A^{-1}A$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

MINOR OF A MATRIX :

Let $A = [a_{ij}]_{m \times n}$ be any matrix.

Determinant of any submatrix of order r is called the minor of order r .

Example:
$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 7 & 9 \\ 3 & 5 & -1 \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} 7 & 9 \\ 5 & -1 \end{vmatrix}$$

$$M_{23} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}$$

Imp**RANK OF A MATRIX**

12.5 to 15 marks

The rank of a given matrix A is said to be r if

- If every minor of A of order $r+1 = 0$
- There is at least one minor of A of order r which does not vanish

$$\rho(A) = r$$

* Normal Form

* Echelon Form

* 3x3 matrix = 2.5 marks

31

Q Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 4 & 10 & 18 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 4 & 10 & 18 \end{vmatrix}$$

$$= 1(90 - 80) - 2(36 - 32) + 3(20 - 20)$$
$$= 10 - 8$$

$$|A| = 2 \neq 0$$

$$\therefore \rho(A) = 3$$

2.5 mark

Q. Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & -3 & -1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & -3 & -1 \end{vmatrix}$$

$$|A| = 1(-3 + 3) - 2(-2 + 2) + 3(-6 + 6)$$

$$|A| = 0$$

$$\therefore |A| = 0$$

$$\therefore \rho(A) < 3$$

There exists a minor of order 2, i.e.,

$$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= 3 - 4$$

$$= -1 \neq 0$$

$$\therefore \rho(A) = 2$$

Mostly asked in exam.

ECHELON FORM OF A MATRIX.

Rank of A , $\rho(A)$ = number of non-zero rows.
(Using Row Transformation)

Q Find the rank of the matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & -3 & -1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & -3 & -1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 + 2R_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ -2 & -3 & -1 \end{bmatrix}$$

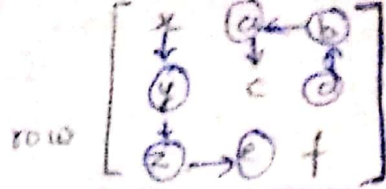
$$R_3 \rightarrow R_3 + R_2$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

By using Echelon form

$$\rho(A) = 2 \text{ (number of non-zero rows)}$$

$$\rho(A) = 2$$



NORMAL FORM OF A MATRIX :

Every non-zero matrix of order $m \times n$ can be reduced by application of elementary row transformation and column transformation into equivalent matrix of one of the form

$$[I_r] \quad \begin{bmatrix} I_r \\ 0 \end{bmatrix} \quad \begin{bmatrix} I_r & 0 \end{bmatrix} \quad \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

where I_r is the identity matrix of order $r \times r$.

These four form are called Normal Form (Canonical form)

Q Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}$ by using normal form.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - 4R_1,$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -3 & -6 \end{bmatrix}$$

$$R_3 \rightarrow \frac{1}{(-3)} R_3$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_2 \rightarrow C_2 - 2C_1, \quad C_3 \rightarrow C_3 - 3C_1$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_3 \rightarrow C_3 - 2C_2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \rightarrow (-1) R_2$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rho(A) = 2.$$

Imp SOLUTION OF THE LINEAR EQUATION

Non-Homogeneous Linear Equations

$$\left. \begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \right\} \text{ are called non-homogeneous linear equations}$$

If all d_1, d_2 & d_3 are equal to zero.
Then, homogeneous linear equation.

Given Equation, In matrix form

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$AX = B$$

$$\left\{ X = A^{-1}B \right.$$

A^* $\rightarrow$ Augmented matrix

$$A^* = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & & & \end{bmatrix}$$

Imp Note:

* If $\rho(A^*) = \rho(A) = \text{number of variables}$
 consistent then system has a unique solution [Whole Number]

* If $\rho(A^*) \neq \rho(A)$
 Then, no solution (Inconsistent)

* If $\rho(A^*) = \rho(A) < \text{number of variables}$
 consistent Then, system has infinitely many solution.
 Ans in = 'k' form

5 marks

Q. ① Solve the equation

$$\begin{aligned} x + y + z &= 6 \\ x + y + z &= 2 \\ 2x + y - z &= 1 \end{aligned}$$

Given Equation in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$AX = B$$

$$\text{Augmented matrix } A^* = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1, \quad R_3 \rightarrow R_3 - 2R_1$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 0 & 0 & -4 \\ 0 & -1 & -3 & -11 \end{bmatrix}$$

$$R_2 \rightarrow \frac{1}{(-2)} R_2$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & -3 & -11 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -3 & -9 \end{bmatrix}$$

$$R_3 \rightarrow \frac{1}{(-3)} R_3$$

$$A^* = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\rho(A^*) = 3 \quad \text{By using Echelon form (No. of non-zero rows)}$$

$$\rho(A) = 3 \quad \text{(No. of non-zero rows)}$$

Number of variables = 3.

$$\therefore \rho(A^*) = \rho(A) = \text{No. of variables.}$$

$\therefore$ Given system has a unique solution.

Solution:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$$

$$x + y + z = 6$$

$$y = 2$$

$$z = 3$$

$$x + 2 + 3 = 6$$

$$x = 1, \quad y = 2, \quad z = 3. \quad \text{Ans}$$

11/10/15

Solve the Equations, $x + y + z = 3$

(1)

$$3x + y - 2z = -2$$

$$2x + 4y + 7z = 7$$

In matrix form,

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$$

$$AX = B$$

$$A^* = \begin{bmatrix} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1, \quad R_3 \rightarrow R_3 - 2R_1$$

$$A^* = \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2$$

$$A^* = \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$\rho(A^*) = 2 \quad (\text{No. of non-zero rows})$$

$$\rho(A) = 2 \quad "$$

$$\text{No. of variables} = 3$$

Since,

$$\rho(A^*) \neq \rho(A)$$

$\therefore$ This system has no solution (inconsistent)

HOMOGENEOUS LINEAR EQUATION:

$$\begin{aligned}a_1 x + b_1 y + c_1 z &= 0 \\a_2 x + b_2 y + c_2 z &= 0 \\a_3 x + b_3 y + c_3 z &= 0\end{aligned}$$

competition

we still use

Note 8 If $\rho(A) = \text{Number of variables}$. Then the given system has a trivial solution (Zero Solution) i.e. $x = y = z = 0$.

II $\rho(A) < \text{No. of variables}$
system has a non-trivial solution (Non-zero Solution)
Infinite solution. {Ans will be in 'k' form}

Q Find the solution of the equation

$$2x_1 - x_2 + x_3 = 0$$

$$3x_1 + 2x_2 + x_3 = 0$$

$$x_1 - 3x_2 + 5x_3 = 0$$

In matrix form,

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 2 & 1 \\ 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A^* = \begin{bmatrix} 2 & -1 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & -3 & 5 & 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$= \begin{bmatrix} 1 & -3 & 5 & 0 \\ 3 & 2 & 1 & 0 \\ 2 & -1 & 1 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1, \quad R_3 \rightarrow R_3 - 2R_1$$

$$= \begin{bmatrix} 1 & -3 & 5 & 0 \\ 0 & 11 & -14 & 0 \\ 0 & 5 & -9 & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2$$

$$= \begin{bmatrix} 1 & -3 & 5 \\ 0 & 11 & -14 \\ 0 & 0 & -54 \end{bmatrix}$$

$$R_3 \rightarrow 19(R_3 - R_2)$$

$$\begin{bmatrix} 1 & -3 & 5 \\ 0 & 11 & -14 \\ 0 & -1 & -4 \end{bmatrix}$$

$$R_3 \rightarrow 11R_3 + R_2$$

$$\begin{bmatrix} 1 & -3 & 5 \\ 0 & 11 & -14 \\ 0 & 0 & -54 \end{bmatrix}$$

$$\rho(A) = \text{No. of variables}$$

This system has trivial solution

$$x_1 = x_2 = x_3 = 0$$

① Solve the equations : $x + 2y + 3z = 0$

$$3x + 4y + 4z = 0.$$

$$7x + 10y + 12z = 0$$

Given equations in matrix form,

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A X = 0$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1, \quad R_3 \rightarrow R_3 - 7R_1$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rho(A) = 3. \quad (\text{No. of non-zero rows})$$

No. of variables = 3. $\Rightarrow \because \rho(A) = \text{no. of variables}$

This system has trivial solution.

$x = y = z = 0$ is the solution of the given equation.

Solve.

(5)

$$\begin{aligned}x - 2y + z - w &= 0 \\x + y - 2z + 3w &= 0 \\4x + y - 5z + 8w &= 0\end{aligned}$$

$$5x - 7y + 2z - w = 0$$

In matrix form,

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = 0$$

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - 4R_1, R_4 \rightarrow R_4 - 5R_1$$

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 9 & -9 & 12 \\ 0 & 17 & -3 & 4 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_2$$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 17 & -3 & 4 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - R_2$$

$$= \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 15 & 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$, No. of variables = 4

$\therefore \rho(A) < \text{No. of variables}$

This system has non-trivial solution

Solution

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x - 2y + z - w \\ 3y - 3z + 4w \\ 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - 2y + z - w = 0$$

$$3y - 3z + 4w = 0$$

$$y = 0$$

$$\text{Let } z = k_1, w = k_2$$

$$x - 2(0) + k_1 - k_2 = 0$$

$$\boxed{x = k_2 - k_1}$$

$$y = 0$$

$$z = k_1$$

$$w = k_2$$

Matrix :

CHARACTERISTICS EQUATION OF A Equation

* Characteristic Matrix of A : The matrix $A - \lambda I$ is called C matrix of A
 $\lambda = \text{scalar}, I = \text{unit matrix}$

* Characteristic Polynomial of A :
 The determinant $|A - \lambda I|$ is called the characteristic polynomial of A.

* Characteristic Equation of A : $|A - \lambda I| = 0$ is called the characteristic Equation of A.

* Characteristic Roots of A : ^(Value) Roots of Equation $|A - \lambda I| = 0$ is called the Characteristic Roots of A

↓ or
Eigen Roots
 ↓ or
Latent Roots

Q. ① Find the eigen values of the matrix

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} \end{aligned}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 4 = 0.$$

$$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0.$$

$$\lambda^2 - 7\lambda + 6 = 0 \quad \rightarrow \text{Characteristic Equation}$$

$$\lambda^2 - 6\lambda - \lambda + 6 = 0.$$

$$\lambda(\lambda - 6) - 1(\lambda - 6) = 0$$

$$(\lambda - 6)(\lambda - 1) = 0$$

$$\boxed{\lambda = 6} \text{ or } \boxed{\lambda = 1}$$

Required characteristic Roots (Values) are
 $\lambda = 1$ or $\lambda = 6$.

Q2) Find the eigen values of the matrix:

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ -\sin \theta & -\cos \theta \end{bmatrix}$$

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} \cos \theta - \lambda & -\sin \theta \\ -\sin \theta & -\cos \theta - \lambda \end{vmatrix} = 0.$$

$$-(\cos \theta - \lambda)(\cos \theta + \lambda) - \sin^2 \theta = 0$$

$$-(\cos^2 \theta - \lambda^2) - \sin^2 \theta = 0$$

$$-\cos^2 \theta + \lambda^2 - \sin^2 \theta = 0$$

$$-1 + \lambda^2 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

Q Find the characteristic roots of the matrix.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

Characteristic equation of A.

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 5 & 2-\lambda & 6 \\ -2 & -1 & -3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda) \left[(2-\lambda)(-3-\lambda) + 6 \right] - 1(-15 - 5\lambda + 12 + 0) + 3(-5 + 4 - 2\lambda) = 0$$

$$(1-\lambda) \left[-6 - 2\lambda + 3\lambda + \lambda^2 + 6 + 15 + 5\lambda - 12 + 3(-1-2\lambda) \right] = 0$$

$$(1-\lambda) \left[\lambda^2 + \lambda + 0 + 5\lambda + 3 - 3 - 6\lambda \right] = 0$$

$$(1-\lambda) \left[\lambda^2 + 0 \right] = 0$$

$$(1-\lambda) \lambda^2 = 0$$

$$\left[\lambda^2 - 6\lambda - 6\lambda - \lambda^3 = \lambda^2 - 5\lambda^2 - 6\lambda^2 \right] = 0$$

$$\lambda^3 = 0$$

$$(1-\lambda) \left[(\lambda^2 + \lambda) - 1(-5\lambda - 3) + 3(-2\lambda - 1) \right] = 0$$

$$\lambda^3 = 0$$

$$\lambda = 0$$

Imp CAYLEY HAMILTON THEOREM...

72

Every square matrix satisfies its own characteristic equation.

Verify Cayley Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Characteristic Eqⁿ for A, $|A - \lambda I| = 0$.

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)(1-\lambda)^2 + 0 \\ &= (1-\lambda)^3 \end{aligned}$$

$$\therefore |A - \lambda I| = 0$$

$$(1-\lambda)^3 = 0$$

$$1 - \lambda^3 - 3\lambda + 3\lambda^2 = 0$$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \quad \text{--- (1)}$$

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

L.H.S.

$\therefore \lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$ $\rightarrow$ from eqⁿ (1)

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-3+3-1 & 0+0+0 & 3-6+3+0 \\ 0+0+0+0 & 1-3+3+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 1-3+3+1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= 0 = \text{R.H.S.}$

$\therefore$ Hence the matrix A satisfies its characteristic equation.

2014

SM

Q. Verify Cayley Hamilton theorem, for the matrix

A = $\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and find A^{-1} .

$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}$

$|A - \lambda I| = -(1-\lambda)(1+\lambda)^2 + 4(2-\lambda)$

$= -(1-\lambda)[-(1+\lambda^2+2\lambda)+4]$

$\Rightarrow (-1-\lambda^2-2\lambda+\lambda+\lambda^3) = (1-\lambda)[- \lambda^2-2\lambda+4]$

$\Rightarrow \lambda^3 + \lambda^2 - 5\lambda + 5 = 0$

$= \lambda^3 + \lambda^2 - 3\lambda + 1$

$\Rightarrow \lambda^3 + \lambda^2 - 5\lambda - 5 = 0$

$$A^2 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

73

$$A \cdot A = \begin{bmatrix} 1+4 & 2-2 & 0 \\ 2-2 & 4+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$|A - \lambda I| = \lambda^3 + \lambda^2 - 5\lambda - 5$$

$$= \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 5 \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore A^3 + A^2 - 5A - 5I = 0.$$

Hence, the matrix A satisfies its characteristic Equation.

$$A^3 + A^2 - 5A = 5I.$$

$$A^3 A^{-1} + A^2 A^{-1} - 5A A^{-1} = 5I A^{-1} \quad (\text{multiplying by } A^{-1})$$

$$A^2 + A - 5I = 5A^{-1}$$

$$5A^{-1} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$5A^{-1} = \begin{bmatrix} 5+1-5 & 0+2-0 & 0+0-0 \\ 0+2-0 & 5-1-5 & 0 \\ 0 & 0 & 1-1-5 \end{bmatrix}$$

$$5A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/5 & 2/5 & 0 \\ 2/5 & -1/5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

How Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$
 Also find A^{-1} . Ans $-\frac{1}{3} \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 1 & 3-\lambda & 5 \\ 1 & 5 & 12-\lambda \end{vmatrix}$$

$$\begin{aligned} &= (1-\lambda)[(3-\lambda)(12-\lambda)-25] - 2(12-\lambda-5) + 3(5-3+\lambda) \\ &= (1-\lambda)[36-3\lambda-12\lambda+\lambda^2-25] - 2(7-\lambda) + 3(2+\lambda) \\ &= 36-3\lambda-12\lambda+\lambda^2-25-14+2\lambda+6+3\lambda \\ &= -\lambda^3 + 13\lambda^2 - 18\lambda - 11 \\ &= (1-\lambda)(\lambda^2 - 15\lambda + 11) - 14 + 2\lambda + 6 + 3\lambda \\ \Rightarrow \lambda^2 - 15\lambda - 11 - \lambda^3 + 15\lambda^2 - 11\lambda + 5\lambda - 8 &= 0 \\ \Rightarrow -\lambda^3 + 16\lambda^2 - 21\lambda + 3 &= 0 \\ \lambda^3 - 16\lambda^2 + 21\lambda - 3 &= 0 \end{aligned}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix} = \begin{bmatrix} 1+2+3 & 2+6+15 & 3+10+36 \\ 1+3+5 & 2+9+25 & 3+15+60 \\ 1+5+12 & 2+15+60 & 3+25+144 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 23 & 49 \\ 9 & 36 & 78 \\ 18 & 77 & 172 \end{bmatrix}$$

74

$$A^3 = A^2 \cdot A \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix} \begin{bmatrix} 6 & 23 & 49 \\ 9 & 36 & 78 \\ 18 & 77 & 172 \end{bmatrix}$$

$$= \begin{bmatrix} 6+18+54 & 23+72+231 & 49+156+516 \\ 6+27+90 & 23+108+385 & 49+234+860 \\ 6+45+216 & 23+180+924 & 49+390+2064 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 78 & 326 & 721 \\ 123 & 516 & 1143 \\ 267 & 1197 & 2503 \end{bmatrix}$$

$$\Rightarrow A^3 - 16A^2 + 21A - 3I$$

$$= \begin{bmatrix} 78 & 326 & 721 \\ 123 & 516 & 1143 \\ 267 & 1197 & 2503 \end{bmatrix} - 16 \begin{bmatrix} 6 & 23 & 49 \\ 9 & 36 & 78 \\ 18 & 77 & 172 \end{bmatrix} + 21 \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 78-96+21-3 & 326-368+42+0 & 721-784+63-0 \\ 123-144+21-0 & 516-576+63-3 & 1143-1248+105-0 \\ 267-288+21-0 & 1197-1232+105+0 & 2503-2752+21-3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\therefore A^3 - 16A^2 + 21A - 3I = 0$$

Hence matrix A satisfies its characteristic eq. A

$$\therefore A^3 - 16A^2 + 21A = 3I$$

$$A^3 A^{-1} - 16A^2 A^{-1} + 21A A^{-1} = 3I A^{-1} \quad [\text{multiplying by } A^{-1}]$$

$$A^2 - 16A + 21I = 3A^{-1}$$

$$3A^{-1} = \begin{bmatrix} 6 & 23 & 49 \\ 9 & 36 & 78 \\ 18 & 77 & 172 \end{bmatrix} - \begin{bmatrix} 16 & 32 & 48 \\ 16 & 48 & 80 \\ 16 & 80 & 192 \end{bmatrix} + \begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix} \quad \underline{\underline{\text{Ans}}}$$

18/Aug/15

5 Marks

EIGEN VECTORS : Latent Vectors or Characteristic Root Vectors.

Let $A = [a_{ij}]$ be a $n \times n$ -square matrix.

Then the eigen vectors X satisfying the equation

$$AX = \lambda_i X$$

or

$$(A - \lambda_i I)X = 0$$

where X is the non-zero matrix vector.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Q. Find the eigen value and eigen vectors of the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Characteristic Eqⁿ of A

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\lambda^2 - 6\lambda - \lambda + 6 = 0$$

$$\lambda(\lambda - 6) - 1(\lambda - 6) = 0$$

$$(\lambda - 1)(\lambda - 6) = 0$$

$$\lambda = 1, 6$$

$\therefore$ Eigen value of the given matrix are 1 and 6.

Eigen Vectors: The eigen vectors $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of A corresponding to the eigen value $\lambda = 1$ is

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad [\lambda = 1]$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad R_1 \rightarrow \frac{R_1}{4}$$

$$R_2 \rightarrow R_2 - R_1$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix} = 0$$

$$x_1 + x_2 = 0$$

$$x_1 = -x_2$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = k$$

$$x_1 = k \quad ; \quad x_2 = -k$$

$$\therefore X = \begin{bmatrix} k \\ -k \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

is the eigen vector corresponding to $\lambda = 1$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 6$.

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$R_2 \rightarrow R_2 + R_1$$

$$\begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -x_1 + 4x_2 \\ 0 \end{bmatrix} = 0$$

$$-x_1 + 4x_2 = 0$$

$$x_1 = 4x_2$$

or

$$5x_2 = 0$$

$$x_2 = 0$$

$$\Rightarrow x_1 = 0$$

$$\frac{x_1}{4} = \frac{x_2}{1} = k$$

$$x_1 = 4k, \quad x_2 = k$$

$$\therefore X = \begin{bmatrix} 4k \\ k \end{bmatrix} \text{ or } \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

is the eigen vector corresponding to $\lambda = 6$.

19/Aug/2015

Find the latent roots and latent vectors of the matrix.

$$A = \begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Characteristic eqn of matrix A,

$$|A - \lambda I| = 0:$$

$$\begin{vmatrix} a-\lambda & h & g \\ 0 & b-\lambda & 0 \\ 0 & 0 & c-\lambda \end{vmatrix} = 0$$

$$(c-\lambda)(a-\lambda)(b-\lambda) - h(0) + g(0) = 0$$

$$\lambda^3 - (a+b+c)\lambda^2 + (ab+bc+ca)\lambda - abc = 0$$

$$\lambda^2 - \lambda(a+b+c) + abc = 0$$

$$(a-\lambda)(b-\lambda)(c-\lambda) = 0$$

$$\lambda = a, \lambda = b, \lambda = c.$$

$\therefore$ Eigen values of given matrix are a, b, c .

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = a$.

$$(A - \lambda I)x = 0.$$

$$\begin{bmatrix} a-a & h & g \\ 0 & b-a & 0 \\ 0 & 0 & c-a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & h & g \\ 0 & b-a & 0 \\ 0 & 0 & c-b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$hx_2 + gx_3 = 0$$

$$\Rightarrow hx_2 = -gx_3$$

$$(b-a)x_2 = 0 \Rightarrow x_2 = 0$$

$$\frac{x_2}{-g} = \frac{x_3}{h} = k$$

$$(c-b)x_3 = 0 \Rightarrow x_3 = 0$$

$$x_2 = -gk, x_3 = hk$$

∴ Eigen vectors corresponding to $\lambda = a$ is

83

$$\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = b$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} a-b & h & g \\ 0 & b-b & 0 \\ 0 & 0 & c-b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a-b & h & g \\ 0 & 0 & 0 \\ 0 & 0 & c-b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (a-b)x_1 + hx_2 + gx_3 = 0 \\ 0 \\ (c-b)x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(a-b)x_1 + hx_2 + gx_3 = 0$$

$$(c-b)x_3 = 0 \Rightarrow x_3 = 0$$

$$\text{Let } x_3 = 0 \quad (a-b)x_1 + hx_2 + g(0) = 0$$

$$(a-b)x_1 = -hx_2$$

$$\frac{x_1}{-h} = \frac{x_2}{a-b} = k$$

$$x_1 = -hk, \quad x_2 = (a-b)k$$

∴ Eigen vectors corresponding to $\lambda = b$ is

$$\begin{bmatrix} -hk \\ (a-b)k \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -h \\ a-b \\ 0 \end{bmatrix}$$

Theorem → A square matrix of order n has latent root $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$. Then A' has also the same latent roots. 87

latent roots.

Proof → let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$

The characteristic Eqⁿ of A : $|A - \lambda I| = 0$

$$= \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} - \lambda & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} - \lambda \end{vmatrix} = 0 \quad \text{--- (I)}$$

$$A' = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \dots & a_{n2} \\ a_{13} & a_{23} & a_{33} & \dots & a_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{nn} \end{bmatrix}$$

Characteristic Eqⁿ of A' : $|A' - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a_{11} - \lambda & a_{21} & a_{31} & \dots & a_{n1} \\ a_{12} & a_{22} - \lambda & a_{32} & \dots & a_{n2} \\ a_{13} & a_{23} & a_{33} - \lambda & \dots & a_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{nn} - \lambda \end{vmatrix} = 0 \quad \text{--- (II)}$$

Value of determinant remains unchanged if rows & columns are interchanged.

∴ From eqⁿ (I) & (II)

$$|A - \lambda I| = |A' - \lambda I| = 0$$

same

Contd.

Hence, the latent roots of A and A' are same.

Theorem: If A is an $n \times n$ triangular matrix, then the elements of the principal diagonals are the characteristic roots of A .

Proof $\rightarrow$ let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$

Characteristic Eqⁿ of A : $|A - \lambda I| = 0$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} - \lambda & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} - \lambda & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$

Expanding along C_1 ,

$$(a_{11} - \lambda) \begin{vmatrix} a_{22} - \lambda & a_{23} & \dots & a_{2n} \\ 0 & a_{33} - \lambda & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} - \lambda \end{vmatrix}$$

$$(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) \dots (a_{nn} - \lambda) = 0$$

$$a_{11} - \lambda = 0, a_{22} - \lambda = 0, a_{33} - \lambda = 0, \dots, a_{nn} - \lambda = 0$$

$$\lambda = a_{11}, a_{22}, a_{33}, \dots, a_{nn}$$

$\therefore$ The latent roots of A are the principal diagonal elements of A .

Imp Theorem : The characteristic roots of a Hermitian matrix are real. $A^{\theta} = A$ 85.

5 Marks

Proof → let A is a Hermitian matrix and λ is a characteristic roots of A and x is a corresponding eigenvectors.
Then, $(A - \lambda I)x = 0$.
 $Ax = \lambda x$ — (i)

Multiplying both sides by x^{θ} in eqⁿ (i).

$$x^{\theta} Ax = x^{\theta} \lambda x$$

$$x^{\theta} Ax = \lambda x^{\theta} x \text{ — (ii)}$$

Taking conjugate transpose both sides of eq — (i)

$$(x^{\theta} Ax)^{\theta} = (\lambda x^{\theta} x)^{\theta}$$

$$(Ax)^{\theta} (x^{\theta})^{\theta} = \bar{\lambda} (x^{\theta} x)^{\theta}$$

$$x^{\theta} A^{\theta} x = \bar{\lambda} (x^{\theta}) (x^{\theta})^{\theta}$$

$$x^{\theta} A^{\theta} x = \bar{\lambda} x^{\theta} x \text{ — (iii)}$$

from (ii) & (iii)

$$\lambda x^{\theta} x = \bar{\lambda} x^{\theta} x$$

$$(\lambda - \bar{\lambda}) x^{\theta} x = 0$$

$$x^{\theta} x \neq 0$$

$$\therefore \lambda - \bar{\lambda} = 0$$

$$\Rightarrow \lambda = \bar{\lambda} \quad (\therefore \lambda \text{ is purely real})$$

$$(A^{\theta})^{\theta} = A$$

$$(AB)^{\theta} = B^{\theta} A^{\theta}$$

$$\therefore (x^{\theta})^{\theta} = x$$

[$\because A$ is a Hermitian matrix]

20/Aug/2015

Theorem^{How}

The characteristic roots of a skew-hermitian are either purely imaginary or zero.

Proof, let A be the skew hermitian matrix and λ is the eigen roots of A . And X be the corresponding eigen vectors

Then, $(A - \lambda I)X = 0$

$$AX = \lambda X \quad \text{--- (i)}$$

Multiplying both sides by X^0 both sides in eqⁿ (i)

$$X^0 AX = X^0 \lambda X$$

$$X^0 AX = \lambda X^0 X \quad \text{--- (ii)}$$

Taking conjugate transpose on both sides in eqⁿ (ii)

$$(X^0 AX)^0 = (\lambda X^0 X)^0$$

$$(AX)^0 (X^0)^0 = \bar{\lambda} (X^0 X)^0$$

$$X^0 A^0 X = \bar{\lambda} X^0 (X^0)^0$$

$$X^0 A^0 X = \bar{\lambda} X^0 X \quad \text{--- (iii)}$$

$$X^0 A^0 X = X^0 (-A) X = \bar{\lambda} X^0 X \quad \text{--- (iv) } [\because A \text{ is a skew Hermitian matrix}]$$

from eqⁿ (ii) & (iii)

$$\lambda X^0 X = -\bar{\lambda} X^0 X$$

$$(\lambda + \bar{\lambda}) X^0 X = 0$$

$$X^0 X \neq 0$$

$$\therefore \lambda + \bar{\lambda} = 0$$

$$\lambda = -\bar{\lambda}$$

$\Rightarrow \therefore \lambda$ is purely imaginary

Theorem The characteristic roots of unitary matrix are of unit modulus.

Proof: Let A be unitary matrix.

$$\therefore AA^{\theta} = I \quad \text{--- (i)}$$

λ be a characteristic roots of A and X is an eigen vector corresponding to λ .

$$\therefore (A - \lambda I)X = 0$$

$$AX = \lambda X \quad \text{--- (ii)}$$

Taking conjugate transpose on both sides in eqⁿ (i)

$$(AX)^{\theta} = (\lambda X)^{\theta}$$

$$X^{\theta} A^{\theta} = \bar{\lambda} X^{\theta} \quad \text{--- (iii)}$$

Multiplying eqⁿ (ii) & (iii)

$$(X^{\theta} A^{\theta})(AX) = (\bar{\lambda} X^{\theta})(\lambda X)$$

$$X^{\theta} (A^{\theta} A) X = (\bar{\lambda} \lambda) (X^{\theta} X)$$

$$X^{\theta} I X = |\lambda|^2 (X^{\theta} X) \quad (\text{Using eqⁿ (i), } \bar{\lambda} \lambda = |\lambda|^2)$$

$$X^{\theta} X = |\lambda|^2 X^{\theta} X$$

$$(1 - |\lambda|^2)(X^{\theta} X) = 0$$

$$(1 - |\lambda|^2) = 0$$

$$[\because X^{\theta} X \neq 0]$$

$$|\lambda|^2 = 1$$

$$|\lambda| = 1$$

$\therefore$ Characteristic roots of A are unit modulus.