UNIT 4: NEWTON'S LAWS OF MOTION AND CONSERVATION PRINCIPLES

STRUCTURE:

4.1 INTRODUCTION

An object is said to be in motion if its position changes with respect to its surroundings in a given time while on the other hand, if the position of the object does not change with respect to its surroundings, it is said to be at rest. A motorbike speeding on road, a bird flying through air, a ship sailing on water, the graceful movements of a dancer are the examples of objects in motion while on the other hand, a pen lying on the table is at rest because its position with respect to the table does not change with time.

To study the motion of an object, you have to study the change in the position of the object with respect to its surroundings. In space, the position of an object is specified by the three coordinates x, y and z. The position of the object changes due to change in one or two or all the three coordinates. The motion of an object is said to be one-dimensional when one of the three coordinates specifying the position of the object changes with time. The motion of a bus on the road, the motion of a train on railway track or an object falling freely under gravity are examples of one-dimensional motion. The motion of an object is said to be two-dimensional when two of the three coordinates specifying the position of the object change with time. Among the well-known examples of two-dimensional motion that you have studied are circular motion and projectile motion. However, your study was limited to the motion along a straight line (one-dimensional motion) and in a two-dimensional plane (two-dimensional motion). But you know that our world is three-dimensional in space. Therefore, we shall begin by studying motion in three dimensions. The motion of an object is said to be three-dimensional when all the three coordinates specifying the position of the object change with time. The motion of a flying kite, gas molecules or the motion of bird in the sky are some examples of three-dimensional motion.

We shall first understand what we mean when we say that an object is moving. We shall learn how to describe the motion of a particle in terms of displacement, velocity and acceleration. In this unit, we shall also study the factors affecting the motion. For this, we shall study Newton's laws of motion and apply them to a variety of situations. We shall use the familiar concept of linear momentum to study the motion of systems having more than one particle and establish the principle of conservation of linear momentum.

4.2 OBJECTIVES

After studying this unit, you should be able to-

- define motion
- apply Newton's laws of motion
- solve problems using Newton's laws of motion
- apply the law of conservation of linear momentum

- apply the impulse of force

4.3 WHAT IS MOTION?

Can you imagine what your life would be like if you were confined to some place, unable to move from one position to another as the time passed? This sentence possess the answer to the question. What is motion? If an object occupies different positions at different instants of time, then we say that it is moving or it is in motion. Thus the study of motion deals with questions- where? And when? Let us recall some definitions relating to motion.

4.3.1 Distance and Displacement

The position of a moving object goes on changing with respect to time. The length of the actual path covered by a body in a time-interval is called ‘distance’ while the difference between the final and the initial positions of an object is known as ‘displacement’. We know that the position of an object is always expressed with respect to some reference point. If the initial position of an object with respect to a reference point is x_1 and after some time it becomes x_2, then the magnitude of the displacement of the object is $x_2 – x_1$.

In your school science courses, you have studied an important difference between ‘distance’ and ‘displacement’.

4.3.2 Speed

The distance travelled by an object in unit interval of time is called the ‘speed’ of the object i.e.

$$\text{speed} = \frac{\text{distance}}{\text{time}-\text{interval}} \quad \text{.....(1)}$$

The speed is represented by ‘v’ and it has unit meter/second. It is a scalar quantity.

4.3.3 Velocity

The displacement of an object in a particular direction in unit interval of time is called the ‘velocity’ of the object i.e.

$$\text{velocity} = \frac{\text{displacement}}{\text{time}-\text{interval}} \quad \text{.....(2)}$$

The velocity is also represented by $\vec{v}$ and its unit is the same as that of speed i.e. meter/second. You know that velocity is a vector quantity.

Let us suppose that an object is moving along a straight line and with respect to some reference point, its position is x_1 at time t_1 and becomes x_2 at time t_2. It means that in time-interval (t_2- t_1), the displacement of the object is (x_2- x_1). Hence, the average velocity of the object during this time-interval would be-

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} \qquad \text{.....(3)}$$

For expressing the difference in a quantity, we use the symbol Δ (delta). Therefore, we can write the average velocity of the object as-

$$\bar{v} = \frac{\Delta\vec{X}}{\Delta t} \qquad \text{.....(4)}$$

If we go on decreasing the time-interval Δt and when Δt becomes infinitesimally small (Δt→0), then from the above formula, we shall be knowing the velocity of the object at a particular instant of time. This velocity is called the 'instantaneous velocity' of the object and is given by

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta x}}{\Delta t} = \frac{\overrightarrow{dx}}{dt} \qquad \text{.....(5)}$$

4.3.4 Acceleration

You should know that if the velocity of a moving object is changing then its motion is known as 'accelerated motion'. It is obvious that change in velocity means the change in magnitude (i.e. speed) or in direction or in both. Thus, the time-rate of change of velocity of an object is called the 'acceleration' of that object, i.e.

$$\text{acceleration} = \frac{\text{change in velocity}}{\text{time}-\text{interval}} \qquad \text{.....(6)}$$

Acceleration is generally represented by 'a' and it has unit meter/second^2. It is also a vector quantity.

Let us suppose that the velocity of a moving object at time t_1 is v_1 and at time t_2, it becomes v_2. It means that in the time-interval (t_2- t_1), the change in the velocity of the object is (v_2- v_1). Therefore, the average acceleration of the object in time-interval (t_2- t_1) is

$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\overrightarrow{\Delta v}}{\Delta t} \qquad \text{.....(7)}$$

If the time-interval Δt is infinitesimally small (i.e. Δt→0), then at a particular time, the instantaneous acceleration is given by

$$a = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta v}}{\Delta t} = \frac{\overrightarrow{dv}}{dt} \qquad \text{.....(8)}$$

If the velocity of an object undergoes equal changes in equal time-intervals, then its acceleration is called 'uniform'

Example 1: The displacement versus time equation of a particle falling freely from rest is given by x= (2.9 ms^{-2}) t^2, where x is in meters, t in seconds. Calculate the average velocity of the particle between t_1= 2 sec and t_2 = 3 sec.

Solution: Here $x= (2.9\ ms^{-2})\ t^2$

At time $t_1 = 2$ sec, $x_1= 2.9 \times (2)^2 = 11.6$ m and at time $t_2= 3$ sec, $x_2 = 2.9 \times (3)^2 = 26.1$ m

Average velocity $\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{26.1-11.6}{3-2} = 14.5$ m/s

Self Assessment Question (SAQ) 1: A particle moves along the x-axis in such a way that its coordinate (x) varies with time (t) according to the expression $x = 2- 6t+ 8t^2$ meter. Find the initial velocity of the particle.

Self Assessment Question (SAQ) 2: Can a body have zero velocity and finite acceleration? Give example.

Self Assessment Question (SAQ) 3: If the displacement of a body is proportional to square of time, state whether the body is moving with uniform velocity or uniform acceleration.

Self Assessment Question (SAQ) 4: Choose the correct option-

The distance covered by a particle as a function of time t is given by $x= 5t^3 + 6\ t^2 -5$. The acceleration of the particle-

(i) remains constant (ii) increases with time (iii) decreases with time (iv) first increases and then decreases with time

4.4 CAUSES OF MOTION

What makes things move? Let us understand this question. The answer of this question was suggested by the great physicist Aristotle, way back in the fourth century B.C. Most of the people believed in his answer that a force which is described as push or pull, was needed to keep something moving. And the motion ceased when the force was removed. This idea made a lot of common sense. But these ideas were first critically examined by Galileo who performed a series of experiments to show that no cause or force is required to maintain the motion of an object.

Let us try to understand some common examples. From our daily experience, we know that the motion of a body is a direct result of its interactions with the other bodies around it which form its environment when a cricketer hits a ball, his or her bat interacts with the ball and modifies its motion. The motion of a freely falling body or of a projectile is the result of its interaction with earth. When an ox stopped pulling an ox-cart, the cart quickly comes to a stop.

An interaction is quantitatively expressed in terms of a concept called 'force'- a push or a pull. When we push or pull a body, we are said to exert a force on it. Earth pulls all bodies towards its centre and is said to exert a force (gravitational) on them. A locomotive exerts a force on the train, it is either pulling or pushing. In this way every force exerted on a body is associated with some other body in the environment.

We should also remember that it is not always that an application of force will cause motion or change motion. For example, we may push a wall i.e. there is an interaction between us and the wall and hence there is a force, but the wall may not move at all.

Thus, force may be described as push or pull, resulting from the interaction between bodies which produces or tends to produce motion or change in motion. The analysis of the relation between force and motion of a body is based on Newton's laws of motion. We will now discuss these laws.

4.4.1 Newton's Laws of Motion

Galileo concluded that any object in motion, if not obstructed, will continue to move with a constant speed along a horizontal line. Therefore, there would be no change in the motion of an object, unless an external agent acted on it to cause the change. That was Galileo's version of inertia. Inertia resists changes, not only from the state of rest, but also from motion with a constant speed along a straight line. Therefore, the interest shifted from the causes of motion to the causes for changes in motion. Galileo's version of inertia was formalized by Newton in a form that has come to be known as Newton's first law of motion.

4.4.2 Newton's First Law

This law states –"**If a body is at rest then it will remain at rest or if it is moving along a straight line with a uniform speed then it will continue to move as such unless an external force is applied on it to change its present state**". This property of bodies showing a reluctance to change their present state is called "inertia". Hence, Newton's first law is also known as the "law of inertia" This law is also called "Galileo law". Newton's first law makes no distinction between a body at rest and one moving with a constant velocity. Both states are 'natural' when no net external force or interaction acts on the body.

4.4.3 Newton's Second Law

Newton's second law tells us what happens to the state of rest or of uniform motion of a body when a net external forces acts on the body i.e. when the body interacts with other surroundings bodies. This law states –"**The time-rate of change of linear momentum of a particle is directly proportional to the force applied on the particle and it takes place in the direction of the force**".

Mathematically,

$$\frac{d\vec{p}}{dt} \propto \vec{F}$$

or $\frac{d\vec{p}}{dt} = k\,\vec{F}$, where $\vec{F}$ is the applied force and k is a constant of proportionality. The differential operator $\frac{d}{dt}$ indicates the time-rate of change.

In MKS or SI system, k = 1. Therefore, $\frac{d\vec{p}}{dt} = \vec{F}$

or simply, $\vec{F} = \frac{d\vec{p}}{dt}$ …..(9)

But linear momentum of the particle $\vec{p} = m\vec{v}$, where m is the mass of the particle and $\vec{v}$ is the velocity of the particle.

Therefore, $\vec{F} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$ (since $\vec{a} = \frac{d\vec{v}}{dt}$ = acceleration of the particle)

Thus, Newton's second law can be written as $\vec{F} = m\vec{a}$ …..(10)

In scalar form, it can be written as F= ma …..(11)

Thus, force is equal to mass times acceleration, if the mass is constant. The force has the same direction as the acceleration. This is an alternative statement of the second law. Newton's second law is also known as 'Law of change in momentum'.

If the position vector of a particle is $\vec{r}$ at a time t then its velocity $\vec{v}$ can be expressed as-

$\vec{v} = \frac{\overrightarrow{dr}}{dt}$ …..(12)

and its acceleration $\vec{a} = \frac{\overrightarrow{dv}}{dt} = \frac{\overrightarrow{d^2r}}{dt^2}$ …..(13)

Therefore, Newton's second law can be written as-

$\vec{F} = m\frac{\overrightarrow{d^2r}}{dt^2}$ …..(14)

If the unit of mass m is kg and the unit of acceleration a is meter/second^2, then the unit of force F is called 'newton'.

If m=1 kg and a= 1 meter/second^2

then magnitude of force F = ma= 1 x 1 = 1 Newton

i.e. 1 newton force is the force which produces an acceleration of 1 meter/second^2 in a body of mass 1 kg.

4.4.4 Newton's Third Law

A force acting on a body arises as a result of its interaction with another body surrounding it. Thus, any single force is only one feature of a mutual interaction between two bodies. We find that whenever one body exerts a force on a second body, the second body always exerts on the first a force which is equal in magnitude but opposite in direction and has the same line of action. A single isolated force is therefore an impossibility.

The two forces involved in every interaction between the bodies are called an 'action' and a 'reaction'. Either force may be considered the 'action' and the other the 'reaction'. This fact is made clear in Newton's third law of motion. This law states-"**To every action there is always an equal and opposite reaction**".

Here the words 'action' and 'reaction' mean forces as defined by the first and second laws.

If a body A exerts a force $\overrightarrow{F_{AB}}$ on a body B, then the body B in turn exerts a force $\overrightarrow{F_{BA}}$ on A, such that

$\overrightarrow{F_{AB}} = -\overrightarrow{F_{BA}}$

So, we have $\overrightarrow{F_{AB}} + \overrightarrow{F_{BA}} = 0$ …..(15)

Notice that Newton's third law deals with two forces, each acting on a different body. This law is also known as 'Law of action-reaction'.

There are two important points regarding Newton's third law. Firstly, we cannot say that this particular force is action and the other one is reaction. Any one may be action and the other reaction. Secondly, action and reaction act on different bodies.

Out of three laws, Newton's second law is most general as first and third law may be derived from second law.

Example 2: A ship of mass 4×10^7 kg initially at rest is pulled by a force of 8×10^4 Newton through a distance of 4 meter. Assuming that the resistance due to water is negligible, calculate the speed of the ship.

Solution: Given mass of the ship m= 4×10^7 kg, Force applied F= 8×10^4 Newton

Using F= ma, the acceleration of the ship a= $\frac{F}{m} = \frac{8 \text{ x } 104}{4 \text{ x } 107} = 2 \times 10^{-3}$ m/sec^2

Distance s = 4 m, initial speed of the ship u= 0

Using equation of motion $v^2 = u^2 + 2as$

$= (0)^2 + 2 \times 2 \times 10^{-3} \times 4 = 16 \times 10^{-3}$

or $v = 0.1265$ m/sec

i.e. speed of the ship = 0.1265 m/sec

Example 3: A satellite in a force-free space sweeps stationary interplanetary dust at a rate $\frac{dm}{dt} = \alpha v$, calculate the acceleration of the satellite.

Solution: Given $\frac{dm}{dt} = \alpha v$

Using Newton's second law of motion $F = \frac{dp}{dt}$

$$= \frac{d(mv)}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$$

Since force F= 0 (for force-free space), therefore $0 = m\frac{dv}{dt} + v\frac{dm}{dt}$

$= ma + v(\alpha v)$ [since $\frac{dv}{dt}$ = a and putting for $\frac{dm}{dt}$]

$= ma + \alpha v^2$

or $a = -\frac{\alpha v^2}{m}$

Thus, the acceleration of the satellite = $-\frac{\alpha v^2}{m}$

Self Assessment Question (SAQ) 5: When a player kicks a football, the football and the player experience forces of the same magnitude but in opposite directions according to Newton's third law. The football moves but the player does not move, Why ?

Self Assessment Question (SAQ) 6: A force produces an acceleration of 18 m/sec^2 in a body of mass 0.5 kg and an acceleration of 6 m/sec^2 in another body. If both the bodies are fastened together then how much acceleration will be produced by this force ?

Self Assessment Question (SAQ) 7: A person sitting in a bus moving with constant velocity along a straight line throws a ball vertically upward. Will the ball return to the hands of the person ? Why?

Self Assessment Question (SAQ) 8: According to Newton's third law, every force is accompanied by an equal and opposite force. How can a movement ever take place?

Self Assessment Question (SAQ) 9: Choose the correct option-

(a) When a constant force is applied to a body, it moves with uniform-

(i) acceleration (ii) velocity (iii) speed (iv) momentum

(b) Inertia is the property by virtue of which the body is-

(i) unable to change by itself the state of rest only

(ii) unable to change by itself the state of uniform linear motion only

(iii) unable to change by itself the direction of motion only

(iv) unable to change by itself the state of rest and of uniform linear motion

Self Assessment Question (SAQ) 10: Fill in the blank-

(i) Newton's first law of motion gives the concept of ……….
(ii) Newton's second law gives a measure of ……………..

4.5 WEIGHT AND MASS

The weight of a body is simply the gravitational force exerted by earth on the body. It is a vector quantity whose direction is the direction of the gravitational force i.e. towards the centre of the earth.

Let us understand the concept of weight in a better way. When a body of mass m falls freely, its acceleration is $\vec{g}$ and the force acting on it is its weight $\vec{w}$. Thus, by using Newton's second law, $\vec{F}$ = m$\vec{a}$, we get

$$\vec{w} = m\vec{g} \qquad .....(16)$$

Since both weight $\vec{w}$ and acceleration due to gravity $\vec{g}$ are directed towards the centre of the earth, we can write w = mg …..(17)

The mass m of a body is an intrinsic property of the body while the weight of a body is different in different localities because acceleration due to gravity g varies from point to point on the earth. The unit of mass is kg while that of weight is kg m/sec^2 or Newton.

4.6 APPLICATIONS OF NEWTON'S LAWS OF MOTION

Newton's laws of motion give us the means to understand most aspects of motion. Let us now apply them to a variety of physical situation involving objects in motion. To apply Newton's laws, we must identify the body whose motion interests us. Then we should identify all the forces acting on the body, draw them in the figure and find the net force acting on the body. Newton's second law can then be used to determine the body's acceleration. Let us use this basic method to solve a few examples.

4.6.1 Projectile Motion

The motion of a bullet fired by a gun and that of a ball thrown by a fieldsman to another are the examples of projectile motion. Let us consider such a particle of mass m. It is thrown from a point O with an initial velocity $\overrightarrow{v_0}$ along OP making an angle θ with the horizontal (Figure 1). Let the particle be at a point Q ($\overrightarrow{OQ} = \vec{r}$) at time t. Neglecting air resistance, let us determine the particle's path. The only force acting on the particle is the weight m$\vec{g}$ of the particle which is constant throughout the motion.

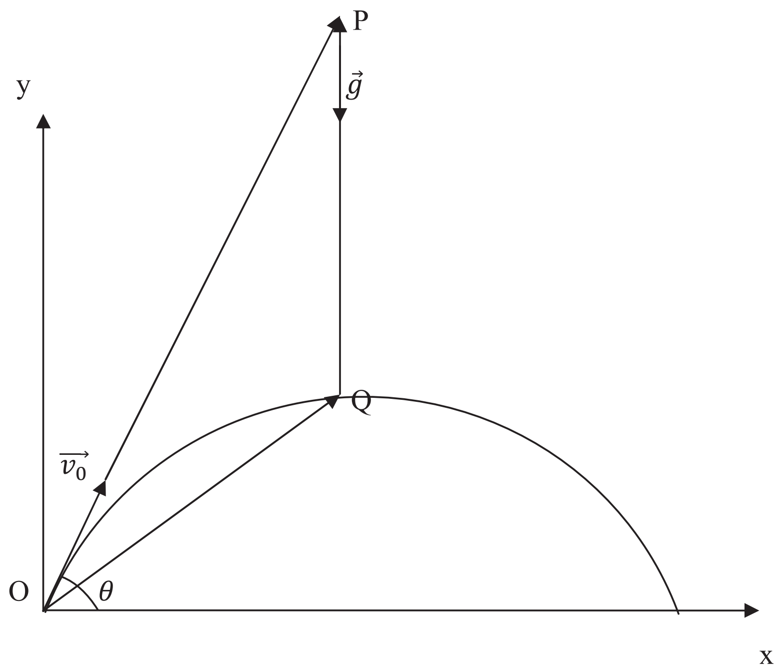


Figure 1

Let us determine the path of the particle. We know by Newton's second law

$\vec{F} = m\overrightarrow{\frac{d^2r}{dt^2}}$ .....(i)

But here $\vec{F} = m\vec{g}$

Therefore, $m\overrightarrow{\frac{d^2r}{dt^2}} = m\vec{g}$ or $\overrightarrow{\frac{d^2r}{dt^2}} = \vec{g}$

or $\frac{d}{dt}(\overrightarrow{\frac{dr}{dt}}) = \vec{g}$ .....(ii)

Integrating with respect to t, we get-

$\frac{\vec{dr}}{dt} = \vec{g}$t + A .....(iii)

where A is the constant of integration.

Applying initial condition, at t= 0, $\frac{\vec{dr}}{dt} = \overrightarrow{v_0}$, therefore, from equation (iii)

$\overrightarrow{v_0} = \vec{g}$ (0) + A

or A= $\overrightarrow{v_0}$

Putting for A in equation (iii), we get,

$\frac{\vec{dr}}{dt} = \vec{g}$t + $\overrightarrow{v_0}$ .....(iv)

Integrating equation (iv) with respect to time t, we get-

$\vec{r} = \overrightarrow{v_0}$ t + $\frac{1}{2}$ $\vec{g}t^2$ + B .....(v)

Where B is a constant of integration.

Applying initial condition, at t = 0, $\vec{r}$ = 0, we get from equation (v)-

0 = $\overrightarrow{v_0}(0) + \frac{1}{2}\, \vec{g}(0)^2$ + B

or B = 0

Putting for B in equation (v), we get-

$\vec{r} = \overrightarrow{v_0}$ t + $\frac{1}{2}$ $\vec{g}t^2$ .....(vi)

We have applied two initial conditions : $\frac{\vec{dr}}{dt} = \overrightarrow{v_0}$ and $\vec{r}$ = 0 at t = 0. Since $\overrightarrow{v_0}$ is along OP and t is scalar, we understand that $\overrightarrow{v_0}$ t is along OP. Again acceleration due to gravity $\vec{g}$ is directed vertically downwards and $\frac{1}{2}$ t^2 is a scalar, therefore $\frac{1}{2}$ $\vec{g}t^2$ is directed vertically downwards i.e. along PQ (Figure 1).

We use the law of vector addition-

$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$ .....(vii)

In this way, we get the location of the particle. As time advances OP is lengthened and therefore, is PQ, and we get the location of the particle by adding $\overrightarrow{OP}$ and $\overrightarrow{PQ}$.

4.6.2 Friction

A heavy block is kept on a horizontal rough floor. You apply a force to pull it but it still does not move. Is it a contradiction of Newton's laws? Let us discuss the motion of the block.

The block is acted upon by two forces-

(i) its weight mg acting vertically downward at its centre of gravity and

(ii) the reactionary-force P exerted on it by the floor directed vertically upward and passes through its centre of gravity.

Since the block is in equilibrium, P= mg. In the figure 2, the lines of action of mg and P are shown slightly displaced for clarity.

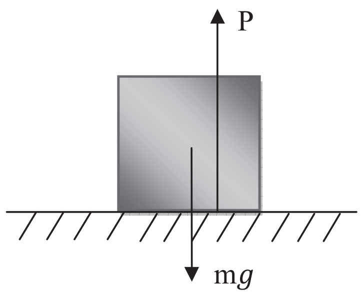
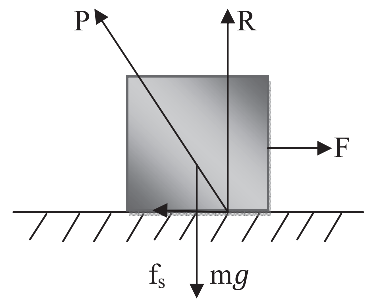


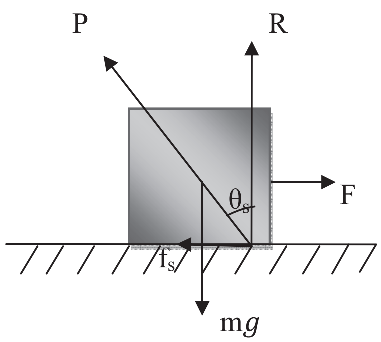
Figure 2



$f_s < \mu_s R$

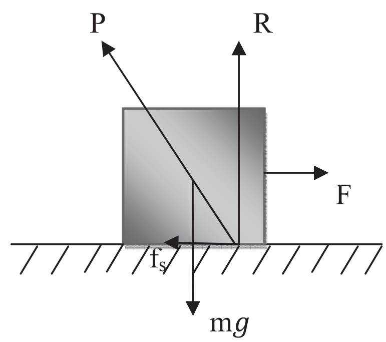
Figure 3

When we apply a small horizontal force F, say towards right (Figure 3), the block does not move. The force P exerted on the block by the floor is now so inclined towards left that P, mg and F may form a closed triangle (since block is still in equilibrium). The force P can be resolved into two components; parallel and perpendicular to the contact-surfaces. The component parallel to the contact surfaces is called the 'force of static friction' f_s which balances the applied force F (F= f_s). The component perpendicular to the contact surfaces is the 'normal reaction' R exerted on the block which balances the weight mg of the block (R= mg).



$f_s = \mu_s R$

Figure 4



$f_k = \mu_k R$

Figure 5

Now, if the applied force F is slightly increased, the block does not still begin to move. This means that the force P is further inclined towards left so that the force of static friction f_s also increases to become equal to the new value of F. Thus, as the applied force F is increased, the force of static friction f_s also increases, but after a certain limit, f_s cannot increase any more. At this moment the block is just to move (Figure 4). This maximum value of the static frictional force f_s is called 'limiting frictional force'(it is equal to the smallest force required to start motion). Now, as the applied force is further increased, the block begins to move.

The limiting (maximum) static frictional force depends upon the nature of the surfaces in contact. It does not depend upon the size or area of the surfaces. For the given surfaces, the limiting frictional force f_s is directly proportional to the normal reaction R i.e.

$f_s \propto R$

or $f_s = \mu_s R$ (for limiting frictional force) …..(i)

where the constant of proportionality μ_s is called the 'coefficient of static friction'. The above formula holds only when f_s has its maximum (limiting) value (Figure 4). Before this stage, $f_s < \mu_s R$ (Figure 3). Hence, usually, $f_s \leq \mu_s R$.

If the direction of the applied force is reversed, the direction of f_s also reverses, while the direction of R remains unchanged. in actual fact, f_s is always opposite to F.

Once the motion starts, the frictional force acting between the surfaces decreases, so that a smaller force F is required to maintain uniform motion (Figure 5). The force acting between the surfaces in relative motion is called the 'dynamic frictional force' f_k which is less than the limiting force of static friction f_s. You know from daily experience that a lesser force is required to maintain the motion of a body than the force required to start the body from rest.

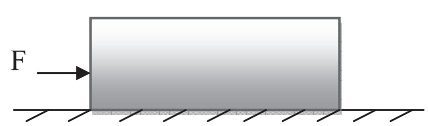
Thus, when the block is in uniform motion, the force of dynamic (or kinetic) friction is

$f_k = \mu_k R$ …..(ii)

where μ_k is the coefficient of dynamic (or kinetic) friction and its value is less than μ_s.

Let us see some examples based on these applications.

Example 4: A block of mass 2 kg is placed on a rough floor. The coefficient of static friction is 0.4. A force F of magnitude 2.5 N is applied on the block, as shown. Calculate the force of friction between the block and floor.



Solution: Given mass of block m= 2 kg, Coefficient of static friction μ_s = 0.4, Force F = 2.5 N

We know the limiting (maximum) force of static friction $f_s = \mu_s R$

$= \mu_s(mg)$ since R= mg

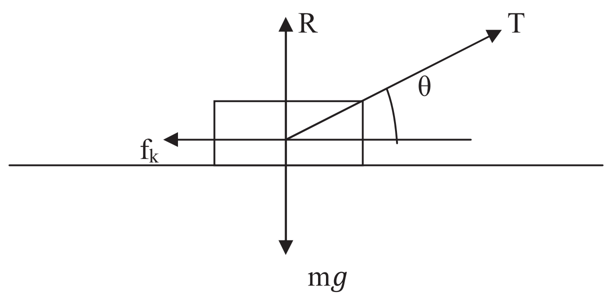
$= 0.4 \times 2 \times 9.8$ (since $g = 9.8$ m/sec^2)

$= 7.84$ N

Obviously, the applied force F is less than the limiting frictional force. Hence, under the force F, the block does not move. Therefore, as long as the block does not move, the (adjustable) frictional force is always equal to the applied force. Thus, the frictional force is 2.5 N.

Example 5: A box of mass m is being pulled across a rough floor by means of a massless rope that makes an angle θ with the horizontal. The coefficient of kinetic friction between the box and the floor is μ_k. What is the tension in the rope when the box moves at a constant velocity?

Solution: Let the tension in the rope be T. All forces acting on the box are shown in the figure.



R is the normal reaction and correspondingly the magnitude of the force of kinetic friction f_k is equal to $\mu_k R$. It is in a direction opposite to the tendency of motion. Since there is no motion in the vertical direction, the resultant of the forces along vertical direction must be zero. Also, as the body moves with a uniform velocity, the resultant force along the horizontal direction is zero. Resolving all the forces along horizontal and vertical direction, we have-

$T \sin\theta + R - mg = 0$ …..(i)

$\mu_k R - T \cos\theta = 0$ …..(ii)

From equation (i), we have-

$R = mg - T \sin\theta$

Putting for R in equation (ii), we get-

$\mu_k (mg - T \sin\theta) - T \cos\theta = 0$

or $\mu_k mg - \mu_k T \sin\theta - T \cos\theta = 0$

or $T = \mu_k mg / (\cos\theta + \mu_k \sin\theta)$

Self Assessment Question (SAQ) 11: Choose the correct option-

(a) The maximum value of static friction is called-

(i) coefficient of static friction (ii) limiting friction (iii) normal reaction (iv) kinetic friction

(b) The limiting friction between two bodies in contact is independent of-

(i) normal reaction (ii) nature of surfaces in contact (iii) the area of surfaces in contact (iv) all of these

(c) The static friction is-

(i) equal to dynamic friction (ii) always less than dynamic friction (iii) always greater than dynamic friction (iv) sometimes greater and sometimes equal to dynamic friction

4.7 LINEAR MOMENTUM

The linear momentum of a particle is defined as the product of the mass of the particle and the velocity of the particle. Usually it is denoted by 'p'.

If m is the mass of a particle and v the velocity then the linear momentum of the particle,

$p= mv$ …..(18)

If kg is the unit of mass and m/sec^2 that of velocity then the unit of linear momentum is given as kg m/sec^2. It is a vector quantity.

In vector form, $\vec{p} = m\vec{v}$ …..(19)

If we have a system of particles of masses $m_1, m_2, m_3, \ldots\ldots m_n$ and velocities $\vec{v_1}, \vec{v_2}, \vec{v_3}, \ldots \ldots \vec{v_n}$, then the total linear momentum of the system would be the vector sum of the momenta of the individual particles i.e.

$\vec{p} = \vec{p_1} + \vec{p_2} + \vec{p_3} + \ldots\ldots\ldots + \vec{p_n}$

$= m_1\vec{v_1} + m_2\vec{v_2} + m_3\vec{v_3} + \ldots\ldots\ldots + m_n\vec{v_n}$

Since $\sum_{i=1}^{n} m_i v_i = M\vec{v_{CM}}$, where M is the total mass and $\vec{v_{CM}}$ is the velocity of the centre of mass of the system.

Thus, $\vec{p} = M\vec{v_{CM}}$ …..(20)

Thus, the total linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass. This suggests that the momentum of the system is the same as if all the mass were concentrated at the centre of mass moving with velocity $\overrightarrow{v_{CM}}$. Hence $\vec{v}$ is known as 'system velocity'.

If the system is isolated so that $\overrightarrow{p}$ is constant, then

$\overrightarrow{v_{CM}}$ = constant

Thus the centre of mass of an isolated system moves with constant velocity, or is at rest.

4.8 CONSERVATION OF LINEAR MOMENTUM

Let us consider a particle of mass m with initial velocity $\overrightarrow{v_1}$. On applying an external force $\vec{F}$ upon the body, its velocity increases to $\overrightarrow{v_2}$ in a time-interval Δt.

The initial linear momentum of the particle, $\overrightarrow{p_1} = m\overrightarrow{v_1}$

Linear momentum of the particle after the time-interval Δt, $\overrightarrow{p_2} = m\overrightarrow{v_2}$

The change in linear momentum in the time-interval Δt is given as-

$\overrightarrow{p_2} - \overrightarrow{p_1} = m\overrightarrow{v_2} - m\overrightarrow{v_1} = m\,(\overrightarrow{v_2} - \overrightarrow{v_1})$

or $\Delta\vec{p} = m\,\Delta\vec{v}$

Dividing both sides by Δt, we have-

$$\frac{\Delta\vec{p}}{\Delta t} = m\,\frac{\Delta\vec{v}}{\Delta t} \qquad \text{.....(21)}$$

In the above equation, $\frac{\Delta\vec{p}}{\Delta t}$ is the rate of change of linear momentum and $\frac{\Delta\vec{v}}{\Delta t}$ is the rate of change of velocity of the particle which is called the acceleration $\vec{a}$.

Therefore, the above equation can be written as-

$$\frac{\Delta\vec{p}}{\Delta t} = m\vec{a} \qquad \text{.....(22)}$$

But $m\vec{a} = \vec{F}$ (Newton's second law)

Therefore, $$\frac{\Delta\vec{p}}{\Delta t} = \vec{F} \qquad \text{.....(23)}$$

Thus, "the rate of change of linear momentum of a particle is equal to the external force applied on the particle and the change in momentum always takes place in the direction of the force". This is an alternative statement of the Newton's second law.

The equation (23) can be written as-

$\frac{d\vec{p}}{dt} = \vec{F}$ …..(24)

If external force $\vec{F}$ = 0, then

$\frac{\Delta\vec{p}}{\Delta t} = 0$

or $\vec{p}$ = constant

i.e. "If the external force acting on a particle is zero, then its linear momentum remains constant". This is known as the 'Principle of Conservation of Linear Momentum'.

For a system of particles, $\overrightarrow{p_1} + \overrightarrow{p_2} + \overrightarrow{p_3} + \ldots\ldots\ldots + \overrightarrow{p_n}$ = constant

4.8.1 Applications of Conservation of Linear Momentum

The conservation of linear momentum governs many phenomena. Few examples are below-

(i) Collisions: When two gross bodies, or two atomic particles collide, the velocities acquired by the bodies after the collision are such that the linear momentum of the system remains conserved. A consideration of conservation of linear momentum shows that in order to slow down the fast neutrons in a reactor, the neutrons should be made to collide with stationary target (nuclei) of nearly the same mass of the neutrons themselves. This is why proton-rich material like paraffin, is a very good moderator.

(ii) Firing of Bullet: Let us consider the firing of bullet from a rifle. Before firing, the linear momentum of the riffle and the bullet is zero. Hence, by conservation of linear momentum, the total momentum of the rifle-bullet system after the firing is also zero. That is, the forward momentum of the bullet is numerically equal to the backward momentum of the rifle.

A rocket works on the principle of conservation of linear momentum.

4.8.2 Newton's Third Law and Conservation of Linear Momentum

When two bodies 1 and 2 (say) collide with each other, then during collision they exert forces on each other. Suppose the exerted force on the body 2 by 1 is $\overrightarrow{F_{12}}$ and that on the body 1 by 2 is $\overrightarrow{F_{21}}$. Suppose, due to these forces, the change in linear momentum of the body 1 is $\Delta\overrightarrow{p_1}$ and that of the body 2 is $\Delta\overrightarrow{p_2}$. Suppose the bodies remain in contact with each other for a time-interval Δt.

During collisions, the bodies are the two parts of one combined body and no external force is acting on this combined body. Hence, by Newton's second law, there should be no change in the momentum of the combined body, i.e.

$\Delta\overrightarrow{p_1} + \Delta\overrightarrow{p_2} = 0$

i.e. $\overrightarrow{p_1} + \overrightarrow{p_2}$ = constant

or $m_1\overrightarrow{v_1} + m_2\overrightarrow{v_2}$ = constant

Differentiating with respect to time t, we get-

$$m_1\frac{d\overrightarrow{v_1}}{dt} + m_2\frac{d\overrightarrow{v_2}}{dt} = 0$$

or $m_1\overrightarrow{a_1} + m_2\overrightarrow{a_2} = 0$

where $\overrightarrow{a_1}$ and $\overrightarrow{a_2}$ are the accelerations of body 1 and 2.

or $\overrightarrow{F_{12}} + \overrightarrow{F_{21}} = 0$

or $\overrightarrow{F_{12}} = -\overrightarrow{F_{21}}$

i.e. the force exerted by the body 1 on the body 2 is equal and opposite to the force exerted by the body 2 on the body 1. This is Newton's third law.

4.9 IMPULSE

You can see many occasions in your daily life when a large force is applied on a body for a short time-interval: for example, hitting a cricket-ball by a bat or a ping-pong ball by a stick, striking a nail by a hammer etc. In such cases, the product of the force and the time-interval is called the 'impulse' of the force.

If a constant force $\vec{F}$ is applied on a body for a short interval of time Δt, then the impulse of this force will be $\vec{F} \times \Delta t$. Impulse is a vector quantity having the direction of force. It may be found by calculating net change in linear momentum.

If $\vec{F}$ is the instantaneous force at time t and it is applied from a time t_1 to a time t_2 , then impulse

$$\vec{I} = \int_{t_1}^{t_2} \vec{F} dt \qquad \text{.....(25)}$$

Let $\overrightarrow{p_1}$ and $\overrightarrow{p_2}$ be initial and final momenta of body.

By Newton's second law, we know that-

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Putting for $\vec{F}$ in equation (25), we get-

$$\vec{I} = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt} dt$$

$= \int_{p_1}^{p_2} \overrightarrow{dp} = [\,\overrightarrow{p}\,]_{p_1}^{p_2}$

$= \overrightarrow{p_2} - \overrightarrow{p_1}$

= net change in momentum

i.e. the change in momentum is equal to the impulse

or $F \times \Delta t = p_2 - p_1$.

The unit of impulse is Newton-sec.

Example 6: A ball of mass 0.35 kg moving horizontally with a velocity 10 m/sec is struck by a bat. The duration of contact is 10^{-3} sec. After leaving the bat, the speed of the ball is 30 m/sec in a direction opposite to its original direction of motion. Calculate the average force exerted by the bat.

Solution: Given mass of the ball m= 0.35 kg, initial velocity of ball v_1= 10 m/sec, final velocity of ball v_2= 30 m/sec, duration of contact $\Delta t = 10^{-3}$ sec

Change in momentum of ball $\Delta p = p_2 - p_1$

$= mv_2 - mv_1 = m(v_2 - v_1)$

$= 0.35[30-(-10)] = 0.35 \times 40$

= 14 kg m/sec

We know that, Impulse= change in momentum

i.e. $F \times \Delta t = \Delta p$

or $F = \Delta p / \Delta t = 14/10^{-3} = 14000$ Newton

Self Assessment Question (SAQ) 12: A body of mass 2 kg is moving with velocity 10 m/sec. Determine its linear momentum.

Self Assessment Question (SAQ) 13: A body is accelerated from rest by applying a force of 30 N. Calculate the linear momentum of the body after 3 sec.

4.10 SUMMARY

In this unit, you have studied about motion and its causes. To present the clear understanding of motion, some basic definitions like distance, displacement, speed, velocity and acceleration have been discussed. You have studied that the motion of a body is a direct result of its interactions with the other bodies around it. You have also studied Newton's laws of motion i.e. Newton's first law (or law of inertia), Newton's second law (law of change in linear momentum) and Newton's third law (law of action and reaction). According to first law " every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it". Newton's second law of motion gives a very important relationship between force and linear momentum and can be expressed as $\vec{F} = \frac{d\vec{p}}{dt}$. For a system of constant mass, it takes the form $\vec{F} = m\vec{a}$. Newton's third law states "to every action there is always an equal and opposite reaction". Forces of action and reaction act on different bodies. In the unit, you have seen that the total linear momentum of a system is conserved if no net external force acts on it which is known as principle or law of conservation of linear momentum. You have also studied the impulse and its relation with linear momentum. Many solved examples are given in the unit to make the concepts clear. To check your progress, self assessment questions (SAQs) are given place to place.

4.11 GLOSSARY

Surroundings – environment, area around a thing or person

Position- location, a place where someone or something is or should be

Specified- particular

Limited- restricted

Confined- restricted

Undergo- suffer

Maintain- sustain

Interactions- exchanges

Resist- refuse to go along with

Friction- resistance

Conservation- protection, preservation or restoration

Assessment- evaluation

4.12 TERMINAL QUESTIONS

1. A ball thrown vertically upward falls at the same place after some time. What is the displacement of the ball?

2. A body is moving on a smooth horizontal surface. Is any force acting on it if it is moving with uniform velocity?

3. The displacement of a particle moving along x-axis is given by $x = a + bt^2$. Find out the acceleration of the particle.

4. A force vector applied on a body is given by $\vec{F} = 2\,\hat{i} - 4\hat{j} + 10\,\hat{k}$ and acquires an acceleration 2 m/sec^2. Find the mass of the body.

5. According to Newton's third law every force is accompanied by an equal and opposite force. How can a movement ever takes place?

6. Calculate the weight of a block of mass 2 kg.

7. A shell is fired from a cannon. The force on the shell is given by $F = 600 - 2 \times 10^5\, t$, where F is in Newton and t in second. The force on the shell becomes zero as soon as it leaves the barrel. Calculate the average impulse imparted to the shell?

8. What is the linear momentum of a 1000 kg car whose velocity is 30 m/sec?

9. The displacement x of a particle moving in one dimension under the action of a constant force is related to time t by equation $t = \sqrt{x} + 5$, where x is in meter and t in second. Find the displacement of the particle when its velocity is zero.

10. An electron starting from rest has a velocity v given by v = At, where A = 3 m/sec^2 and t is the time. What will be the distance covered by the body in first 2 sec.

11. Mention the difference between distance and displacement.

12. What are Newton's laws of motion? Explain.

13. What is meant by impulse of a force? Prove that the impulse of a force is equal to the change in linear momentum. Give the unit of impulse.

14. What is the principle of conservation of linear momentum? Derive from it Newton's third law of motion.

15. Write notes on-

    (i) Linear momentum (ii) Impulse (iii) Friction

4.13 ANSWERS

Self Assessment Questions (SAQs):

1. Given x = 2- 6t+ 8t^2 meter

Differentiating with respect to time t we get-

$v = \frac{dx}{dt} = 0- 6+16\ t$

$= -6 + 16\ t$

For initial velocity of the particle, t = 0

Therefore, initial velocity of the particle v = -6 + 16 (0)

= - 6 m/sec

2. Yes, the body can have zero velocity and finite acceleration. For example, at the highest point of a body thrown vertically upward, the body has zero velocity and acceleration = g

3. Given, displacement of a body ∝ $time^2$

i.e. $x \propto t^2$

or $x = A\ t^2$, where A is a constant

Velocity $v = \frac{dx}{dt} = 2A\ t$

Obviously, the velocity depends on time.

Acceleration $a = \frac{dv}{dt} = 2A$

Obviously, the acceleration of the body is uniform.

4. Given $x= 5t^3 + 6\ t^2 -5$

Differentiating with respect to time t, we get-

$v = \frac{dx}{dt} = 15\ t^2 + 12\ t$

Acceleration of the particle $a = \frac{dv}{dt} = 30\ t + 12$

Obviously, the acceleration of the particle increases. Therefore, the correct option is (ii).

5. The reaction force acts on the player. Due to large mass (inertia) of the player, the force is not able to make him move.

6. Let F be the force.

$m_1 = 0.5$ kg, $a_1 = 18$ m/sec^2, $a_2 = 6$ m/sec^2

$F = m_1 a_1 = 0.5 \times 18 = 9$ N

Again, $F = m_2 a_2$

or $m_2 = \frac{F}{a_2} = \frac{9}{6} = 1.5$ kg

If both the bodies are fastened together, then total mass $M = m_1 + m_2$

$= 0.5 + 1.5 = 2$ kg

Using $F = M a$

or $a = \frac{F}{M} = \frac{9}{2} = 4.5$ m/sec^2

7. Yes, the ball will return to the hands of the person. The reason is that due to inertia the horizontal velocity of ball remains equal to the velocity of bus.

8. The motion becomes possible since action and reaction, though act simultaneously but on different bodies.

9. (a) (i) , (b) (iv)

10. (i) inertia (ii) force

11. (a) (ii), (b) (iii), (c) (iii)

12. Mass of the body m=2 kg, v = 10 m/sec

Linear momentum $p = mv = 2 \times 10 = 20$ kg m/sec

13. F= 30 N, initial velocity $v_1 = 0$ (since body is at rest initially), $\Delta t = 3$ sec

We know, $F \times \Delta t = p_2 - p_1$

or $F \times \Delta t = p_2 - mv_1$

$30 \times 3 = p_2 - m(0)$

or $90 = p_2$

or $p_2 = 90$

i.e. linear momentum of the body after 3 sec is 90 kg m/sec

Terminal Questions:

1. Zero
2. No
3. Given $x = a + bt^2$

   Differentiating with respect to t

   $\frac{dx}{dt} = 0 + 2\,bt$

   Again differentiating with respect to time

   $\frac{d^2x}{dt^2} = 2\,b$

   i.e. acceleration a = 2 b

4. Given $\vec{F} = 2\,\hat{i} - 4\hat{j} + 10\,\hat{k}$, acceleration a = 2 m/sec^2

   Magnitude of the force F= $\left|2\,\hat{i} - 4\hat{j} + 10\,\hat{k}\right| = \sqrt{(2)^2 + (4)^2 + (10)^2}$

   $= \sqrt{4 + 16 + 100} = \sqrt{120} = 10.95$ N

   Now using F = ma

   $m = \frac{F}{a} = \frac{10.95}{2} = 5.48$ kg

5. The motion becomes possible since action and reaction, though act simultaneously but on different bodies.

6. Mass m = 2 kg, g = 9.8 m/sec^2

   Weight w = mg

   = 2 x 9.8 = 19.6 kg m/sec^2

7. $F = 600 - 2 \times 10^5\, t$

   Force F = 0 in time t given by

   $0 = 600 - 2 \times 10^5\, t$

   or $t = \frac{600}{2 \times 10^5} = 3 \times 10^{-3}$ sec

Impulse of force I = $\int_0^t F\,dt$

$= \int_0^t (600 - 2 \times 10^5\, t)\,dt$

= 600 t - $\frac{2\times10^5 t^2}{2}$ = 600 t – 10^5 t^2

= 600 x 3 x 10^{-3}- 10^5 x (3 x 10^{-3})^2

=0.9 N-sec

8. Mass of car m = 1000 kg, velocity of car v = 30 m/sec

Linear momentum of car p = mv

= 1000 x 30 = 3 x 10^4 kg m/sec

9. Given t = $\sqrt{x}$ + 3

or $\sqrt{x}$ = t – 3

Squaring both sides, x = (t – 3)^2

or x = t^2 + 9 - 6 t

or x = t^2 – 6t + 9

Differentiating both sides with respect to t, we get-

$\frac{dx}{dt}$ = 2 t – 6

or v = 2 t – 6

when v = 0, 2 t – 6 =0

or t = 3 sec

At t = 3 sec, displacement x = (t – 3)^2

or x = (3 – 3)^2

= 0

Hence displacement of particle is zero when its velocity is zero.

10. Given v = At, where A = 3 m/sec^2

v = $\frac{dx}{dt}$ = At

or $dx = At\,dt$

Integrating both sides, we get

$\int_0^x dx = A \int_0^t t\,dt$

or $x = A\,t^2 / 2$

$= 3 \times (2)^2 / 2$

$= 6$ meter

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UNIT 5: PRINCIPLES OF CONSERVATION OF ENERGY AND ANGULAR MOMENTUM.

STRUCTURE:

5.1 INTRODUCTION

In the previous unit we have studied about the motion, causes of motion, Newton's laws of motion and their applications. We have also studied the important law of conservation of linear momentum and its applications. We have also gone into impulse and its relation with change in linear momentum. We often feel that when we execute a motion, some energy is spent and sometimes we say that work is done at the cost of some energy. We know that energy is a very important physical quantity. A dancing, running person is said to be more energetic in comparison of a sleeping, snoring man. In Physics, a moving particle is said to have more energy compared to an identical particle at rest. In the present unit, you will learn about energy and its different kinds in details. We will also study the very important principles of conservation of energy and angular momentum. These principles have very wide applications and are used ordinarily in your Physics courses.

5.2 OBJECTIVES

After studying this unit, you should be able to-

- Compute work done by forces
- apply work-energy theorem
- distinguish between conservative and non-conservative forces
- apply principle of conservation of energy
- solve problems based on conservation of energy
- compute power in various mechanical systems
- solve problems based on conservation of angular momentum

5.3 WORK

The word 'work' has a special meaning in Physics. If a teacher stands near a table and delivers a lecture for some times, then according to the principles of Physics, no work is done. In Physics, work is said to be done when an external force acting on a particle displaces it. The work done by the force on the particle is defined as " the scalar product of the force and the displacement".

If the force $\vec{F}$ displaces a particle by a displacement $\vec{r}$, then the work done by the force $\vec{F}$ is given as-

$$W = \vec{F}.\,\vec{r} = F\, r \cos\theta \qquad \ldots..(1)$$

where θ is the angle between force $\vec{F}$ and displacement $\vec{r}$.

or work = force x displacement along the direction of force

(r cosθ) is the magnitude of projection of $\vec{r}$ on the force vector $\vec{F}$.

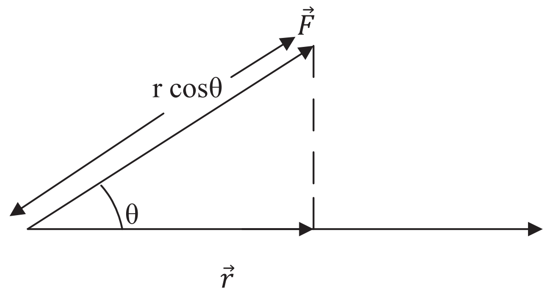


Figure 1

Thus, the work done by a force on a body is defined as the product of magnitudes of force and the component of displacement in the direction of force.

Let us consider a particle P moving along path AB under a force $\vec{F}$ (Figure 2) and this force displaces the particle through an infinitesimal displacement $\overrightarrow{dr}$ then the work done by the force

$$dW = \vec{F}.\overrightarrow{dr} \qquad .....(2)$$

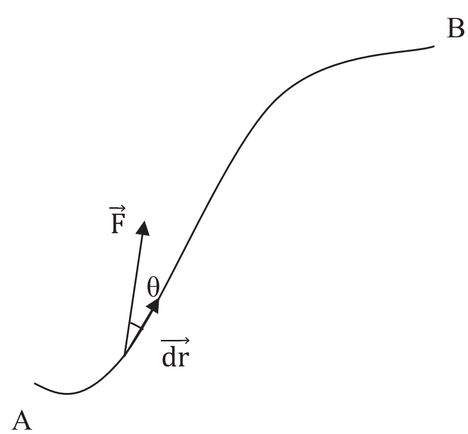


Figure 2

Therefore, total work done by the force in displacing the particle from A to B is given as-

$$W = \int_A^B dW = \int_A^B \vec{F}.\overrightarrow{dr} \qquad .....(3)$$

or $$W = \int_A^B (F\, dr\, \cos\theta) \qquad .....(4)$$

where θ is the angle between $\vec{F}$ and $\overrightarrow{dr}$. Obviously the work done depends on the magnitudes of force, displacement and the angle between them.

Let us discuss different cases for work done.

If $\theta = 0^0$ i.e. the force vector and displacement vector are parallel to each other, then work done $W = \vec{F}.\vec{r} = F\, r \cos\theta$

$= F\, r \cos 0^0 = Fr$

If $\theta = 90^0$ i.e. the displacement of the particle is right angle to the force, then the work done

$W = \vec{F}.\vec{r} = F\, r \cos\theta$

$= F\, r \cos 90^0 = F\, r\, (0) = 0$

It means that if a person holding a heavy weight in his hand moves along a level floor, he does no work since the vertical supporting force of his hand is at right angles to the direction of motion. Similarly, you know that when a satellite revolves around the earth, the direction of the force applied by the earth (centripetal force) is always perpendicular to the direction of motion of the satellite. Hence no work is done on the satellite by the centripetal force i.e. centripetal force acting on a body moving in a circle does no work because the force is always at right angles to the direction of motion. Thus the work done in a circular motion is always zero.

You know that if a particle is freely falling vertically then the force of gravity ($m\vec{g}$) acts on the particle vertically downward and the displacement of the particle is also vertically downward. In this case, $\theta = 0^0$, then work done $W = \vec{F}.\vec{r} = F\, r \cos\theta$

$= F\, r \cos 0^0 = F\, r$

If force and displacement are opposite in direction i.e. $\theta = 180^0$, then work done

$W = \vec{F}.\vec{r} = F\, r \cos\theta$

$= F\, r \cos 180^0 = -\, F\, r$

Thus, you see that if the force is in the same direction as the displacement, the work is positive. If it is opposite to the displacement, the work is negative. Thus, when a person lifts a body from the ground, the work done by the lifting force (upward) of his hand is positive but the work done by the gravitational force (which acts downward) is negative. But on the other hand, when the person lowers the body to ground, the work done by the upward force of his hand is negative but that by the gravitational force is positive. Similarly, when a body slides on a fixed surface, the work done by the frictional force exerted on the body is negative since this force is always opposite to the displacement of the body.

If displacement r = 0, then work done W = F r cosθ = F (0) cosθ = 0 i.e. if the displacement of the particle is zero, then the work done is zero. It means that if the force causes no displacement, the work is zero. A stationary (standing) person holding a heavy weight in his hand may become tired in the physiological sense but according to the principles of Physics, he is not doing any work. Again, the man who has tired to move a luggage but failed, has not done any work because although he has exerted force but the displacement remains zero. As a teacher standing near a table and delivering a lecture, does not do any work according to the principle of Physics.

If the force is varying, then work may be calculated graphically. If we draw a graph between force F and displacement r, then work done by the force F during displacement from r to (r + dr) is given by

$$W = \int_{r_1}^{r_2} F\, dr$$

= Area enclosed by F- r curve

= Area enclosed PQRS (Figure 3)

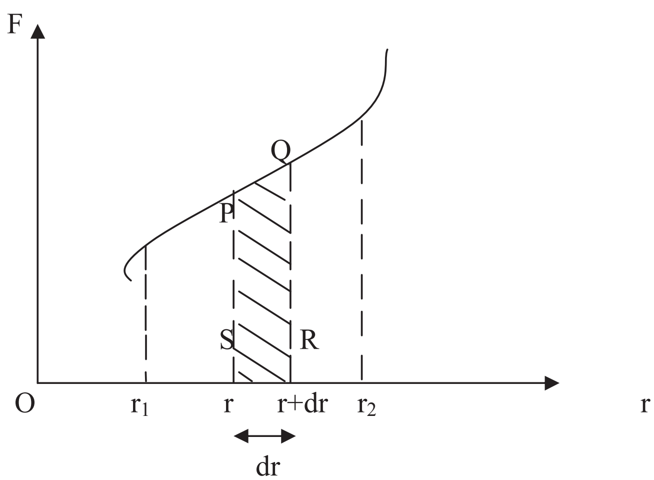


Figure 3

The unit of work is Joule (J). Its other unit is erg.

1 Joule = 10^7 erg

Work is a scalar quantity.

5.3.1 Work in stretching a spring

Let us consider a situation in which one end of a spring is fixed and the other end to a block which can move on a horizontal plane.

(a) (b)

Figure 4

Let x = 0 denotes the initial position of the block when the spring is in its original (natural) length (Figure 4 a). Now the block moves from x = 0 to $x = x_1$ by the application of force F (Figure 4 b). We shall calculate the work done on the block in moving from x = 0 to $x = x_1$.

When the spring is stretched slowly, the stretching force increases steadily as the spring elongates i.e. force is variable. When the spring (or block) stretched through a distance $x = x_1$ by applying a force F at the block, the spring on account of its elasticity, exerts a restoring force which acts in the opposite direction of displacement according to Hooke's law of elasticity.

Thus F = - k x …..(5)

where 'k' is called force constant of the spring. The negative sign indicates that the force and the displacement are opposite in direction.

The work done during a small displacement dx can be written as-

$dW = \vec{F}.\overrightarrow{dx}$

$= F\,dx\cos 180^0 = -F\,dx$

$dW = -(-k\,x)\,dx$ (putting for F)

$dW = k\,x\,dx$

The total work done as the block is displaced from x = 0 to $x = x_1$ is

$$W = \int_{x=0}^{x=x_1} kxdx = k\left[\frac{x^2}{2}\right]_{x=0}^{x=x_1}$$

$= \frac{1}{2} k x_1^2$ …..(6)

If the block moves from $x = x_1$ to $x = x_2$, then total work done will be

$W = \int_{x=x_1}^{x=x_2} kxdx = k\left[\frac{x^2}{2}\right]_{x=x_1}^{x=x_2}$

or $W = \frac{1}{2} k(x_2^2 - x_1^2)$ …..(7)

We should note that if the block is displaced from $x = x_1$ to $x = x_2$ and brought back to $x = x_1$, the work done by the spring force is zero.

Example 1: A block of mass 200 gm is displaced by a distance of 2 m by the application of a force of magnitude 3 N at 30^0. Calculate the work done.

Solution: Given mass of the block m = 200 gm = 0.2 Kg, Displacement r = 2 meter

Force F = 3 N, Angle $\theta = 30^0$

Work done $W = \vec{F}.\vec{r} = F\ r\ \cos\theta$

$= 3 \times 2 \times \cos 30^0 = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$ Joule = 5.19 Joule

Example 2: Calculate the work done in pulling a spring by 10 cm. The force constant of the spring is 500 N/m.

Solution: Given Increase in length of the spring x_1 = 10 cm = 0.1 meter

Force constant of the spring k = 500 N/m

Work done in pulling the spring $W = \frac{1}{2} k\, x^2$

$= \frac{1}{2} \times 500 \times (0.1)^2 = 250 \times 0.01 = 2.5$ Joule

Example 3: A force F = (10 + 0.2 x) acts on a body in the x-direction, where F is in Newton and x in meter. Find out the work done by this force during a displacement from x = 0 to x = 2 m.

Solution: Force F = (10 + 0.2 x) N, Displacement from x = 0 to x = 2 m, $\theta = 0^0$

Work done $W = \int_{x=0}^{x=2} \vec{F}.\overrightarrow{dx}$

$= \int_{x=0}^{x=2} Fdx\cos\theta = \int_{x=0}^{x=2} Fdx\cos 0^0$

$= \int_{x=0}^{x=2} Fdx = \int_{x=0}^{x=2} (10 + 0.2\,x)\, dx = \left[10\,x + 0.2\frac{x^2}{2}\right]_0^2$

$= [10\,x + 0.1x^2]_0^2 = \{10 \times 2 + 0.1(2^2)\} - \{10 \times 0 + 0.1(0^2)\}$

= 20.4 Joule

Self Assessment Question (SAQ) 1: A coolie carries a box on his head on a level platform from one place to another. Estimate the work done by the coolie.

Self Assessment Question (SAQ) 2: A car moves with a uniform speed on a smooth level road. Neglecting air resistance, find out the work done by the car.

Self Assessment Question (SAQ) 3: A particle of mass m is moving in a circle of radius r with uniform speed v. What is the work done in a complete revolution? In half revolution?

5.4 POWER

"The time-rate of doing work by an agent or a machine is called power".

Power = $\frac{\text{Work}}{\text{Time}}$

If W is the work done by an agent in t second, then his power P is given as-

$P = \frac{W}{t} = \frac{\Delta W}{\Delta t}$ …..(8)

Instantaneous power $P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t}$

$= \frac{dW}{dt} = \frac{d}{dt}(\vec{F}.\vec{r})$

Since $W = \vec{F}.\vec{r}$

Therefore, $P = \vec{F}.(\frac{\overrightarrow{dr}}{dt})$ (if force $\vec{F}$ is constant)

Since $\frac{\overrightarrow{dr}}{dt} = \vec{v}$ (velocity)

Therefore, $P = \vec{F}.\vec{v}$ …..(9)

This is the expression for P in the form of scalar product of $\vec{F}$ and $\vec{v}$.

The unit of power is Joule/sec or Watt (W). The other popular units of Power are Horse Power (HP) and Kilo Watt (KW)

1 HP = 746 W and 1KW = 1000 W

If the work done W= 1 Joule and time t = 1 sec

Then power $P = \frac{W}{t} = \frac{1\text{ Joule}}{1\text{ sec}} = 1$ W

i.e. if 1 Joule of work is done in 1 sec, then power is 1 Watt.

The power of a normal person is from 0.05 HP to 0.1 HP.

Example 4: A machine does 50 J work in 2 sec. What is its power?

Solution: Given, W = 50 J, t = 2 sec

Power $P = \frac{W}{t} = \frac{50\text{ Joule}}{2\text{ sec}} = 25$ W

Self Assessment Question (SAQ) 4: A body is moved from rest, along a straight line by a machine delivering constant power. Calculate the distance moved by the body as a function of time t.

5.5 ENERGY

"The capacity of a body to do work is called its energy". The energy is always measured by the work the body is capable of doing. Therefore, the unit of energy is the same as that of work i.e. Joule.

Energy has various forms such as mechanical energy, heat energy, sound energy, chemical energy, light energy, magnetic energy etc. In this unit, we shall concentrate on mechanical energy which includes kinetic energy and potential energy.

5.5.1 Kinetic Energy

"The energy possessed by a body by virtue of its motion is called kinetic energy (K.E.)" i.e. the kinetic energy of a moving body is measured by the amount of work done in bringing the body from the rest position to its present position or which the body can do in going from its present position to the rest position.

Suppose a body of mass m is initially at rest. When we apply a constant force F on the body, it starts moving under an acceleration, then by Newton's second law, we have

F = ma

or $a = \frac{F}{m}$

Suppose, the body acquires a velocity v in time t in moving a distance x.

Using third equation of motion $v^2 = u^2 + 2as$, we have-

$v^2 = 0 + 2ax$ (here u = 0, s = x)

or $v^2 = 2\frac{F}{m}x$

or $\frac{1}{2}m v^2 = F x$

But Fx is the work done by force F on the body in moving it a distance x. Due to this work the body has itself acquired the capacity of doing work. This is the measure of the kinetic energy of the body. Hence if we represent kinetic energy of a body by K, then

$K = F x = \frac{1}{2}m v^2$

or $K = \frac{1}{2}m v^2$ .....(10)

Kinetic energy = $\frac{1}{2}$ × mass × speed^2

Thus the kinetic energy of a moving body is equal to half the product of the mass (m) of the body and the square of its speed (v^2). We see that in this formula v occurs in the second power and so the speed has a larger effect, compared to mass, on the kinetic energy.

We can write equation (10) as-

$K = \frac{1}{2}m(\vec{v}.\vec{v})$ .....(11)

The kinetic energy of a system of particles is the sum of the kinetic energies of all its constituent particles i.e.

$K = \sum_i \frac{1}{2}m_i v_i^2$

If a body is initially moving with a uniform speed u and on applying force F on it, its speed increases from u to v in a distance x, then again using third equation of motion $v^2 = u^2 + 2$ as, we have-

$v^2 = u^2 + 2\,ax$

or $v^2 - u^2 = 2\,ax$

$= 2\left(\frac{F}{m}\right)x$ (since $a = \frac{F}{m}$)

or $F x = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

F x is the work done W on the body by the force. Therefore,

$W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$ .....(12)

($\frac{1}{2}$ mv^2 - $\frac{1}{2}$ mu^2) is the increase in the kinetic energy of the body. Thus when a force acts upon a moving body, then the kinetic energy of the body increases and this increase in kinetic energy is equal to the work done.

The unit of kinetic energy is Kg m^2/sec^2 or Joule. It is a scalar quantity like work.

5.5.2 Potential Energy

A body can do work also by virtue of their position or state of strain. **"The energy possessed by a body by virtue of its position or state of strain is called the potential energy (PE) of the body"**. For example, the water at the top of a water-fall can rotate a turbine when falling on it. The water has this capacity by virtue of its position (at a height). Similarly, a wound clock-spring keeps the clock running by virtue of its state of strain. Thus water and wound spring both have potential energy- the former has gravitational potential energy and the later has elastic potential energy.

The potential energy of a body depends on reference level chosen for zero potential energy.

5.5.3 Gravitational Potential Energy

Let us consider that a body of mass m is raised to a height h from earth's surface. In this process work is done against the force of gravity (mg). This work is started in the body in the form of gravitational potential energy U.

This potential energy U = Work done against force of gravity = Weight of the body × height

$$= mg \times h = mgh$$

or $U = mgh$ …..(13)

If this body falls on the earth, an amount mgh of work may be obtained from it.

5.5.4 Elastic Potential Energy

When a spring is stretched or compressed, work has to be done due to the elasticity of the spring. This work is stored as the potential energy of the body which is given by-

$$U = \frac{1}{2} kx^2 \quad \text{…..(14)}$$

where k is force constant of the spring and x is increase in length.

If spring is already stretched by amount x_1 and is further stretched by amount x_2 then work done in stretching the spring from x_1 to x_2 is

$$W= \frac{1}{2} k (x_1 + x_2)^2 - \frac{1}{2} kx_1^2$$

$= \frac{1}{2} k (x_1^2 + x_2^2 + 2x_1x_2 – x_1^2)$

$= \frac{1}{2} kx_2 (x_2 + 2x_1)$

Therefore, increase in potential energy $\Delta U = W = \frac{1}{2} kx_2 (x_2 + 2x_1)$ …..(15)

Example 5: A spring obeys Hooke's law with a force constant 800 N/m. If it is stretched through 10 cm, how much work is required in this process?

Solution: Given, Force constant k = 800 N/m, x = 10 cm = 0.1 m

Required work $W = \frac{1}{2} kx^2$

$= \frac{1}{2} \times 800 \times (0.1)^2 = 4$ Joule

Example 6: A man ascends to a temple from ground level a vertical rise of 1,800 meter. His mass is 50 Kg. He takes 6 hours. What is the average power exerted?

Solution: Given, h= 1,800 meter, m= 50 Kg , t = 6 hours = 6 × 60 × 60 = 21,600 sec

$P = \frac{W}{t} = \frac{mgh}{t} = \frac{50 \times 9.8 \times 1800}{21600} = 40.83$ W

Example 7: A car of mass 200 Kg is running with a uniform speed of 80 Km/hour. Calculate its kinetic energy.

Solution: Given, mass of car m = 200 Kg, $v = 80$ Km/Hour $= \frac{80 \times 1000}{60 \times 60} = 22.22$ m/sec

Kinetic energy $K = \frac{1}{2} m v^2$

$= \frac{1}{2} \times 200 \times (22.22)^2 = 49372.84$ Joule

Self Assessment Question (SAQ) 5: What happens to the mechanical energy which is spent in raising a heavy body from a lower to a higher level?

Self Assessment Question (SAQ) 6: Does the work done in raising a box onto a platform depend on how fast is it raised?

Self Assessment Question (SAQ) 7: Establish the relation between kinetic energy and linear momentum.

Self Assessment Question (SAQ) 8: Choose the correct option-

(i) Of the following the one possessing the kinetic energy is-

(a) water stored in a dam (b) a bullet in flight (c) stretched rubber band

(d) air in bicycle pump

(ii) The kinetic energy of a moving body varies with mass directly-

(a) m^{-1} (b) m^2 (c) m (d) m^0

(iii) The kinetic energy of a body of mass 1 Kg and momentum 4 N-sec is-

(a) 8 J (b) 7 J (c) 16 J (d) 0 J

5.6 WORK-ENERGY THEOREM

Work-Energy theorem states that **"the work done by the net force acting on a body is equal to the change in kinetic energy of a body".** i.e.

Work = gain in kinetic energy

$$= \frac{1}{2} mv_2^2 - \frac{1}{2} mv_1^2$$

Let us consider a body of mass m acted upon a net force F along x-axis. If body moves from a position x_1 to position x_2 along the x-axis, its velocity increases from v_1 and v_2. The work done by the force in this displacement is –

$$W = \int_{x_1}^{x_2} F\, dx \qquad \text{.....(16)}$$

By Newton's second law, we know

$$F = ma = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx} \qquad \left(\text{Putting } \frac{dx}{dt} = v\right)$$

Therefore, $W = \int_{x_1}^{x_2} mv\frac{dv}{dx}dx$

$$= m\int_{x_1}^{x_2} v\frac{dv}{dx}dx = m\int_{v_1}^{v_2} vdv$$

$$= m\left[\frac{v^2}{2}\right]_{v_1}^{v_2} = \frac{1}{2} mv_2^2 - \frac{1}{2} mv_1^2$$

or $W = \frac{1}{2} mv_2^2 - \frac{1}{2} mv_1^2$ .....(17)

or $W = K_2 - K_1$ .....(18)

where K_1 and K_2 are the initial and final kinetic energies of the body. Thus, if ΔK represents the change in kinetic energy, $\Delta K = K_2 - K_1$ then, we have-

$$W = \Delta K \qquad .....(19)$$

This is the mathematical statement of work-energy theorem.

5.7 CONSERVATIVE AND NON-CONSERVATIVE FORCES

We divide the forces in two categories- conservative forces and non-conservative forces. We can define these forces in various ways.

"A force acting on a particle is conservative if the particles after going through a complete round trip, returns to its initial position with the same kinetic energy as it had initially". Let us understand this with some examples.

When we throw a ball upward against the gravity of earth, the ball reaches a certain height coming momentarily to rest so that its kinetic energy becomes zero, then it returns to our hand under gravity with same kinetic energy with which it was thrown (assuming air resistance to be zero). Thus the force of gravity is conservative force.

Similarly, elastic force exerted by an ideal spring is conservative. The electrostatic force is also conservative.

"A force acting on a particle is non-conservative if the particle, after going through a complete round trip, returns to its initial position with changed kinetic energy".

In the above example, we have assumed the air-resistance to be zero. Actually, air-resistance (viscous force) is always there. This force always opposes the motion. Therefore, a part of kinetic energy is always spent in overcoming this force. Hence, the ball returns to hand with smaller kinetic energy than it had initially. Obviously, this viscous force which is responsible for the decrease in kinetic energy is non-conservative force. Similarly, frictional force between two bodies or planes is a non-conservative force. The force of induction is also an example of non-conservative force.

We can also distinguish between conservative and non-conservative forces in terms of work done. **"A force acting on a particle is conservative if the net work done by the force in a complete round trip of the particle is zero, if the net work done is not zero, then the force is non-conservative".**

Yet, there is a third way of distinguishing between conservative and non-conservative forces. Suppose a particle acted upon by a conservative force goes from P to Q along path 1 and returns to P along path 2 (Figure 5 a). As the force is conservative, the work done in the outgoing journey is equal and opposite to that in the return journey the work done in complete round trip which is equal to the net gain in kinetic energy is zero as defined earlier. Hence i.e. $W_{P\rightarrow1\rightarrow Q} = -W_{Q\rightarrow2\rightarrow P}$ …..(20)

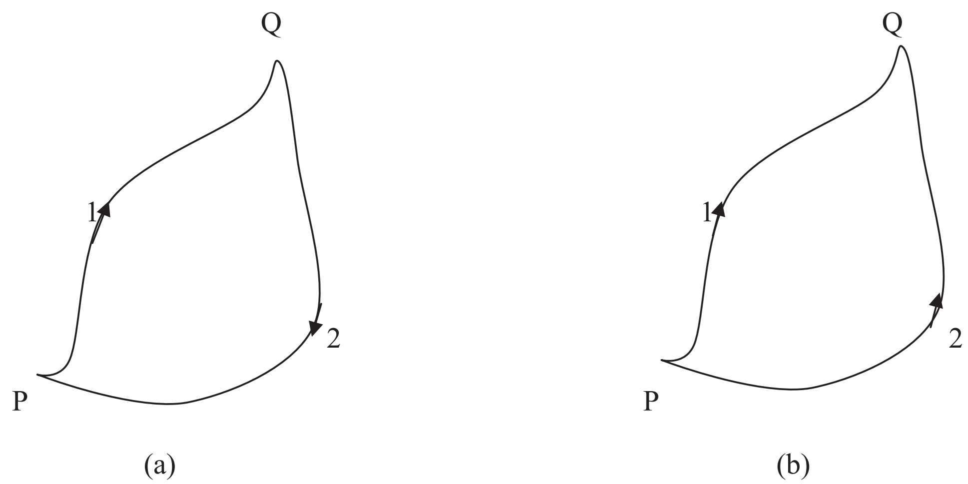


Figure 5

If the particle be moved from P to Q along the path 2 (Figure 5 b), the work done would be equal and opposite to that in moving from Q to P along the path 2 i.e.

$W_{P\rightarrow2\rightarrow Q} = - W_{Q\rightarrow2\rightarrow P}$ …..(21)

Comparing equations (20) and (21), we get-

$W_{P\rightarrow1\rightarrow Q} = W_{P\rightarrow2\rightarrow Q}$ …..(22)

This shows that the work done by the conservative force in moving the particle from P to Q along the path 1 is the same as that along the path 2. Thus if the work done by a force depends only on the initial and final states and not on the path taken, it is called a conservative force.

5.8 CONSERVATION OF ENERGY

"If a system is acted on by conservative forces, the total mechanical energy of the system remains constant". This is called the principle of conservation of energy.

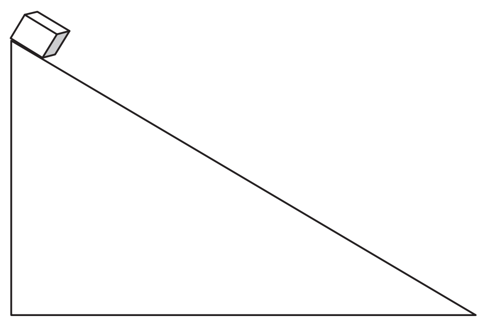
i.e. Mechanical Energy E = Kinetic Energy (K.E.) + Potential Energy (P.E.)

= Constant

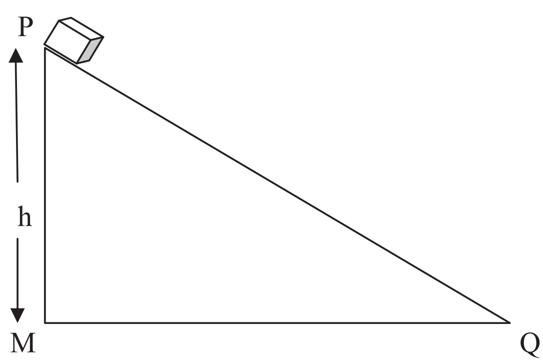
or $E = K + U =$ Constant (under conservative force) …..(23)

The total mechanical energy (K + U) is not constant if non-conservative forces such as friction, act between the parts of the system. We cannot apply the principle of conservation of energy in presence of non-conservative forces and a more general law stated as “The total energy of the universe remains constant” holds. This simply means that the energy may be transformed from one form to another. For example, in loudspeakers, electric bells; the electrical energy is converted into sound energy while in electromagnet, electrical energy is converted into magnetic energy and for a ball falling on earth, the mechanical energy is converted into heat energy etc. The work-energy theorem is still valid even in the presence of non-conservative forces.

Example 8: A block of mass m slides along a frictionless surface as shown in figure. If it is released from rest at P, what is its speed at Q?



Solution:



Let v be the speed of block at Q.

Applying principle of conservation of energy, Total mechanical energy at P = Total mechanical energy at Q

Kinetic energy at P + Potential energy at P = Kinetic energy at Q + Potential energy at Q

Considering MQ as reference level –

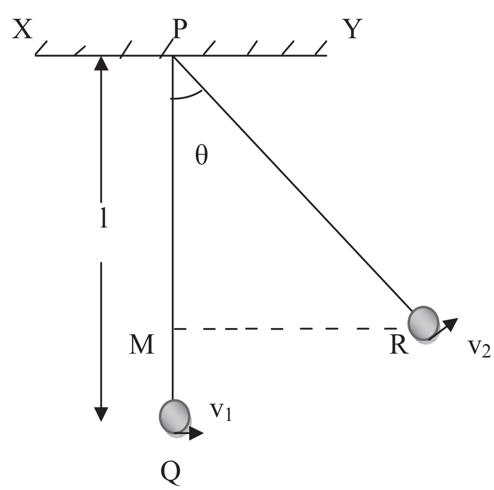
$\frac{1}{2}$ m $(0)^2$ + mgh = $\frac{1}{2}$ m v^2 + 0

mgh = $\frac{1}{2}$ m v^2

v = $\sqrt{2gh}$

Example 9: A pendulum bob has a speed 3 m/sec while passing through its lowest position. What is its speed when it makes an angle of 60^0 with the vertical? The length of the pendulum is 0.5 m. Take g = 10 m/sec^2

Solution:



Let v_2 be the speed of the bob when it makes an angle of 60^0 with the vertical.

Applying principle of conservation of energy-

Total mechanical energy at Q = Total mechanical energy at R

Kinetic energy at Q + Potential energy at Q = Kinetic energy at R + Potential energy at R

Considering XY as reference level –

$\frac{1}{2}$ mv_1^2 + mgl = $\frac{1}{2}$ mv_2^2 + mg (PM) …..(i)

In right angle triangle PMR-

PM = PR cosθ

= l cosθ

Putting for PM in equation (i), we get-

$\frac{1}{2} mv_1^2 + mgl = \frac{1}{2} mv_2^2 + mg \ (l \cos\theta)$

$\frac{1}{2} v_1^2 + gl = \frac{1}{2} v_2^2 + g \ (l \cos\theta)$

or $v_1^2 + 2\ gl = v_2^2 + 2g\ l \cos\theta$

or $v_2^2 = v_1^2 + 2\ gl - 2g\ l \cos\theta$

or $v_2 = \sqrt{v_1^2 + 2\ gl - 2g\ l \cos\theta}$

Here v_1 = 3 m/sec, l = 0.5 m, g = 10 m/sec^2, θ = 60^0

Therefore, $v^2 = \sqrt{3^2 + 2 \times 10 \times 0.5 - 2 \times 10 \times 0.5 \cos 60^0}$

$= \sqrt{9 + 10 - 10 \times \frac{1}{2}} \quad = \sqrt{19 - 5}$

$= \sqrt{14}$ = 3.74 m/sec

Self Assessment Question (SAQ) 9: If energy is neither created nor destroyed, what happens to the so much energy spent against friction?

Self Assessment Question (SAQ) 10: What happens to the mechanical energy which is spent in raising a heavy body from a lower to a higher level?

5.9 ANGULAR MOMENTUM

"The moment of linear momentum of a particle rotating about an axis is called angular momentum of the particle". It is denoted by J.

If a particle be rotating about an axis of rotation, then

J = linear momentum × distance

= p × r

= mv× r (since p = mv)

or J = mvr

where m, v and r are the mass of the particle, linear velocity and distance of particle from axis of rotation respectively.

But $v = r\omega$, where ω is the angular velocity

Therefore, $J = m(r\omega)r = mr^2\omega$

$$\text{or} \quad J = I\omega \qquad \text{.....(24)}$$

where $mr^2 = I$ = Moment of Inertia of particle about the axis of rotation

Let us suppose a body be rotating about an axis with an angular velocity ω. All the particles of the body will have the same angular velocity ω but different linear velocities.

Let a particle be at a distance r_1 from the axis of rotation, the linear velocity of this particle is given by-

$v_1 = r_1\omega$

If m_1 be the mass of the particle, then its linear momentum $p_1 = m_1v_1$

The moment of this momentum about the axis of rotation i.e. angular momentum of the particle J_1 = linear momentum × distance

$= p_1 \times r_1$

$= m_1v_1 \times r_1$

$= m_1(r_1\omega) \times r_1$

$= m_1r_1^2\,\omega$

Similarly, if the masses of other particles be $m_2, m_3, \ldots$ and their respective distances from the axis of rotation be $r_2, r_3, \ldots$, then the moments of their linear momenta about the axis of rotation will be $m_2r_2^2\,\omega$, $m_3r_3^2\,\omega$, ...respectively. The sum of these moments of linear momenta of all the particles i.e. the angular momentum of the body is given by-

$J = m_1r_1^2\,\omega + m_2r_2^2\,\omega + m_3r_3^2\,\omega + \ldots\ldots$

$= (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots)\omega$

$= (\sum mr^2)\omega$

$$\text{or} \quad J = (\sum mr^2)\omega \qquad \text{.....(25)}$$

But $(\sum mr^2) = I$, is the moment of inertia of the body about the axis of rotation and plays the same role in rational motion as mass plays in linear motion. This will be discussed in unit 6.

Therefore, angular momentum J = I ω .....(26)

The unit of angular momentum is Kg-m^2/sec. It is a vector quantity.

In vector form, $\vec{J} = \vec{r} \times \vec{p}$ .....(27)

or $\vec{J} = r\,p\,\sin\theta\,\hat{n}$ .....(28)

where θ is the angle between $\vec{r}$ and $\vec{p}$ and n^ is the unit vector perpendicular to the plane containing $\vec{r}$ and $\vec{p}$.

Magnitude of angular momentum J = r p sinθ .....(29)

5.10 CONSERVATION OF ANGULAR MOMENTUM

We know that relation for angular momentum-

J = I ω .....(30)

The rate of change of angular momentum-

$$\frac{\Delta J}{\Delta t} = I\,\frac{\Delta \omega}{\Delta t}$$

$$= I\,\alpha \qquad .....(31)$$

(since $\frac{\Delta\omega}{\Delta t} = \alpha$, angular acceleration)

But I α = τ (Torque), in analogy with Newton's law for linear motion Ma= F(Force).

Therefore, $\frac{\Delta J}{\Delta t} = \tau$ .....(32)

i.e. the time-rate of change of angular momentum of a body is equal to the external torque acting upon the body. Actually, a torque is required for rational motion just as a force is needed to cause a linear motion.

If τ = 0, then $\frac{\Delta J}{\Delta t} = 0$

or ΔJ = 0

or J = Constant

or I ω = Constant .....(33)

i.e. **"If no external torque is acting upon a body rotating about an axis, then the angular momentum of the body remains constant"**. This is called the law of conservation of angular momentum. If I decreases, ω increases and vice-versa.

In vector form-

$\vec{J} = \vec{r} \times \vec{p}$

Differentiating both sides with respect to time t, we get-

$$\frac{d\vec{J}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$= \left(\frac{d\vec{r}}{dt} \times \vec{p}\right) + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right)$$

$$= (\vec{v} \times m\vec{v}) + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right) \qquad \left(\text{since } \frac{d\vec{r}}{dt} = \vec{v} \text{ and } \vec{p} = m\vec{v}\right)$$

$$= m(\vec{v} \times \vec{v}) + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right)$$

$$= 0 + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right) \qquad (\text{since } \vec{v} \times \vec{v} = 0)$$

or $\frac{d\vec{J}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$ …..(34)

By Newton's second law, $\frac{d\vec{p}}{dt} = \vec{F}$

Therefore, $\frac{d\vec{J}}{dt} = \vec{r} \times \vec{F}$ …..(35)

But $\vec{r} \times \vec{F} = \vec{\tau}$, the torque acting on the particle

Therefore, equation (35) becomes-

$\frac{d\vec{J}}{dt} = \vec{\tau}$ …..(36)

i.e. the time-rate of change of angular momentum of a particle is equal to the torque acting on the particle.

If $\vec{\tau} = 0$, then $\frac{d\vec{J}}{dt} = 0$

i.e. $\vec{J}$ = Constant …..(37)

Let us discuss some examples based on conservation of angular momentum.

When a diver jumps into water from a height, he does not keep his body straight but pulls in his arms and legs towards the centre of his body. On doing so the moment of inertia I of his body decreases. Since the angular momentum I ω remains constant, therefore, on decreasing I, his angular velocity ω correspondingly increases. Hence jumping he can rotate his body in the air.

Suppose a man with his arms outstretched and holding heavy dumb-bells in each hand, is standing at the centre of a rotating table. When the man pulls in his arms, the speed of rotation of the table increases. This is why? The reason is that on pulling in the arms, the distance of the dumbbells from the axis of rotation decreases and therefore, the moment of inertia I of the man decreases. But according to conservation of angular momentum, the total angular momentum remains constant. Therefore, on decreasing moment of inertia I, the angular velocity ω increases.

Example 10: A mass of 3 Kg is rotating on a circular path of radius 1.0 m with angular velocity of 40 radian/sec. If the radius of the path becomes 0.8 m, what will be the value of angular velocity?

Solution: Given, m = 3 Kg, r_1 = 1.0 m, ω_1 = 40 radian/sec, r_2= 0.8 m

Using law of conservation of angular momentum-

$I_1 \omega_1 = I_2 \omega_2$

or $(mr_1^2)\, \omega_1 = (mr_2^2)\, \omega_2$

or $r_1^2\, \omega_1 = r_2^2\, \omega_2$

$(1)^2\, (40) = (0.8)^2\, \omega_2$

or ω_2 = 62.5 radian/sec

Self Assessment Question (SAQ) 11: If the earth suddenly contracts to half its radius, what would be the length of the day? By how much would the duration of day be decreased?

Self Assessment Question (SAQ) 12: Choose the correct option-

(i) Angular momentum of a body is the product of-

(a) linear velocity and angular velocity (b) mass and angular velocity

(c) centripetal force and radius (d) moment of inertia and angular velocity

(ii) When a torque acting on a system is zero, what is conserved-

(a) angular velocity (b) linear momentum (c) force (d) angular momentum

5.11 SUMMARY

In the present unit, we have studied about work, power, energy, work-energy theorem, conservative and non-conservative forces, conservation of energy and angular momentum. We have proved that work done on a particle is equal to the change in its kinetic energy. We have also studied the differences between conservative and non-conservative forces. A force is called conservative if the work done by it during a round trip of a system is always zero otherwise non-conservative. The force of gravitation, electrostatic force, force by a spring etc. are conservative forces while friction is an example of non-conservative force. In this unit, we have concentrated on mechanical energy. In the unit, we have studied the principle of conservation of energy according to which, the total mechanical energy of a system is conserved if the system is acted on by conservative forces. We have also focused on angular momentum and its conservation with few examples. You have seen that if there is no external torque acted on a system then the angular momentum of the system is conserved. To make the concepts more clear, many solved examples are incorporated in the unit. To check your progress, self assessment questions (SAQs) are given in the unit.

5.12 GLOSSARY

Execute – perform a skilful action

Energy – the strength and vitality required to keep active

Conservation- preservation or restoration of the natural environment

Distinguish – recognize or treat as different

External- relating to the outside of something

Displace- move from the proper or usual position

Centripetal – moving towards a centre

Exert- apply or bring to bear a force

Stretch- be able to be made longer or wider without tearing or breaking

Mechanical- relating to or operated by a machine or machinery

5.13 TERMINAL QUESTIONS

1. Is it possible that a body be in accelerated motion under a force acting on the body yet no work is being done by the force? Give example.

2. A particle moves along x-axis from x = 0 to x = 4 m under the influence of a force $F = (5 - 2x + 3x^2)$ N. Calculate the work done in this process.

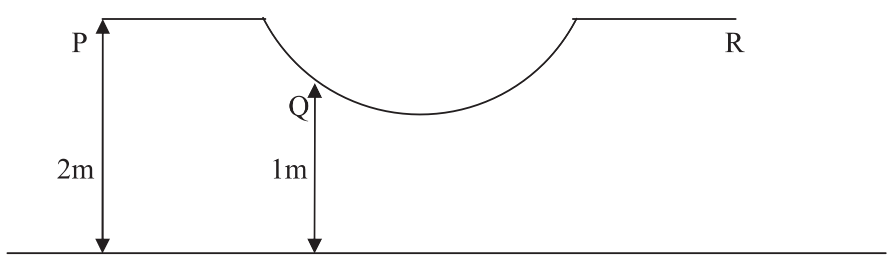
3. A block is pushed through 4 m across a floor offering 50 N resistance. How much work is done by the resisting force?

4. Calculate the work done by a force $F = kx^2$ acting on a particle at an angle of 60^0 with x-axis to displace it from 1 m to 3 m along the x-axis.

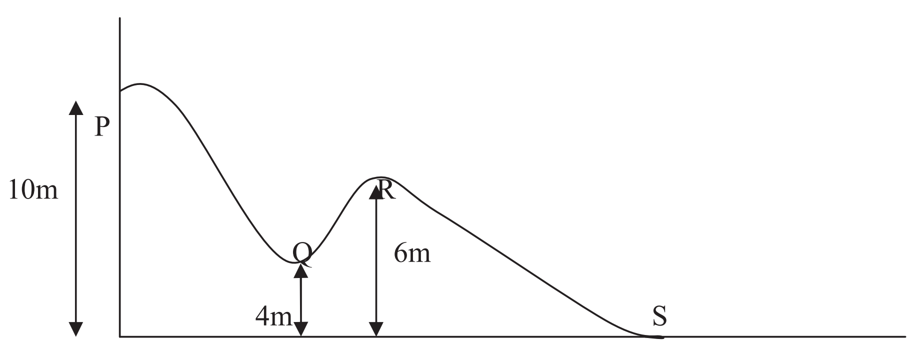
5. If the mass of a body is reduced to half and its velocity is doubled, then what will be ratio of kinetic energy?

6. A boy whose mass is 51 Kg climbs with constant speed, a vertical rope 6 m long in 10 sec. How much work does the boy perform? What is his power output during the climb?

7. A particle is placed at the point P of a frictionless track PQR as shown in figure. It is pushed slightly towards right. Find its speed when it reaches the point Q. Take g = 10 m/sec^2



8. The given figure shows the vertical section of a frictionless surface. A block of mass 2 Kg is released from position P. Compute its kinetic energy as it reaches positions Q, R and S.



9. A ball tied to a string takes 2 sec in one complete revolution in a horizontal circle. If by pulling the cord, the radius of the circle is reduced to half of the previous value, then how much time the ball will now take in one revolution?
10. What is the meaning of work in Physics? What should be the angle between the force and the displacement for maximum and minimum work?
11. Prove and discuss work-energy theorem.
12. Define and explain the difference between conservative and non-conservative force.
13. What is energy? Discuss the different types of mechanical energy with examples.
14. What is the principle of conservation of energy?
15. Define angular momentum. How is the angular momentum of a body conserved? Explain.

5.14 ANSWERS

Self Assessment Questions (SAQs):

1. According to principle of Physics, work = force × displacement along the direction of force. The weight of the box acts vertically downward and the displacement is horizontal i.e. the angle between force of gravity and displacement is 90^0, therefore, $W = Fr\cos\theta = Fr\cos 90^0 = Fr(0) = 0$, i.e. the coolie does no work.
2. The weight of the car (i.e. force of gravity) and displacement of car are at right angle i.e. $\theta = 90^0$, therefore work done $W = Fr\cos\theta = Fr\cos 90^0 = 0$, i.e. work done by car is zero.
3. We know that in circular motion the displacement and force (centripetal force) are at right angle always, therefore the work done by the centripetal force is always zero.
4. Given, P = Constant i.e. $P = Fv$ Constant

   But $F = ma$, therefore, $P = (ma)v$

   or $P = mav$ …..(i)

   Using first equation of motion $v = u + at$

   $v = 0 + at = at$ (since $u = 0$)

   $v = at$

   Therefore, $P = (ma)at = ma^2t$

   or $a^2t = \frac{P}{m}$ = Constant

or a^2t = Constant .....(ii)

Using second equation of motion $s = ut + \frac{1}{2}at^2$

$s = 0\times t + \frac{1}{2}at^2$

$s = \frac{1}{2}at^2$

Squaring both sides, $s^2 = \frac{1}{4}a^2t^4$

or $s^2 = \frac{1}{4}(a^2t)t^3$

or $s^2 \propto t^3$ (since a^2t is constant)

5. The mechanical energy spent in raising a heavy body from a lower to a higher level is not lost but is stored in the form of potential energy.

6. No, since work done W = mgh

7. We know, $K = \frac{1}{2}mv^2$

Multiplying by m in numerator and denominator of RHS, we get-

$K = \frac{1}{2}\frac{m^2v^2}{m}$

or $K = \frac{1}{2}\frac{(mv)^2}{m} = \frac{p^2}{2m}$

8. (i) (b), (ii) (c), (iii) (a)

9. The energy is dissipated in the form of heat. The heat energy so produced is not available for work.

10. By the law of conservation of angular momentum-

$I\omega$ = Constant

or $I_1\omega_1 = I_2\omega_2$

Here, $I_1 = \frac{2}{5}MR_1^2$, $\omega_1 = \frac{2\pi}{T_1} = \frac{2\pi}{24} = \frac{\pi}{12}$ radian/hour, $I_2 = \frac{2}{5}MR_2^2 = \frac{2}{5}M(\frac{R_1}{2})^2 = \frac{MR_1^2}{10}$, $\omega_2 = \frac{2\pi}{T_2}$

Therefore, $(\frac{2}{5}MR_1^2)(\frac{\pi}{12}) = (\frac{MR_1^2}{10})(\frac{2\pi}{T_2})$

or T_2 = 6 hours

Decrease in duration of day = 24 – 6 = 18 hours

11. (i) (d), (ii) (d)

Terminal Questions:

1. Yes, it is possible in a uniform circular motion of a body since instantaneous velocity is always perpendicular to force. For example, moon revolves about the earth under centripetal force and gets accelerated but work done is zero.

2. Given, Force F = $(5 - 2x + 3x^2)$ N

Work done $W = \int_{x=0}^{x=4} \vec{F}.\overrightarrow{dx}$

$$= \int_{x=0}^{x=4} F dx \cos\theta = \int_{x=0}^{x=4} F dx \cos 0^0 \qquad (\text{since } \theta = 0^0)$$

$$= \int_0^4 F dx = \int_0^4 (5 - 2x + 3x^2)\, dx = \left[5x - \frac{2x^2}{2} + \frac{3x^3}{3}\right]_0^4$$

= 68 N

3. Given, r = 4 m, F = 50 N, Obviously here $\theta = 0^0$

Work done $W = \vec{F}.\vec{r} = Fr\cos\theta$

$$= 50 \times 4 \cos 0^0 = 200 \text{ Joules}$$

4. Given $F = kx^2$, $\theta = 60^0$, $x_1 = 1$ m, $x_2 = 3$ m

We know $W = \int_{r_1}^{r_2} \vec{F}.\overrightarrow{dx} = \int_{r_1}^{r_2} F dx \cos\theta$

$$= \int_{x_1=1}^{x_2=3} kx^2 \cos 60^0 dx = \int_1^3 (kx^2)\frac{1}{2} dx$$

$$= \frac{k}{2}\int_1^3 x^2 dx = \frac{k}{2}\left[\frac{x^3}{3}\right]_1^3 = 4.33\text{ k}$$

5. Let $m_1 = m$, $v_1 = v$

Hence $m_2 = m/2$, $v_2 = 2v$

$K_1 = ½(m_1v_1^2)$, $K_2 = ½ (m_2v_2^2)$

Ratio of kinetic energies $K_1/K_2 = ½(m_1v_1^2)/ ½ (m_2v_2^2)$

$$= m_1v_1^2/ m_2v_2^2 = mv^2/\{(m/2)(2v)^2\} = ½$$

i.e. kinetic energy becomes doubled.

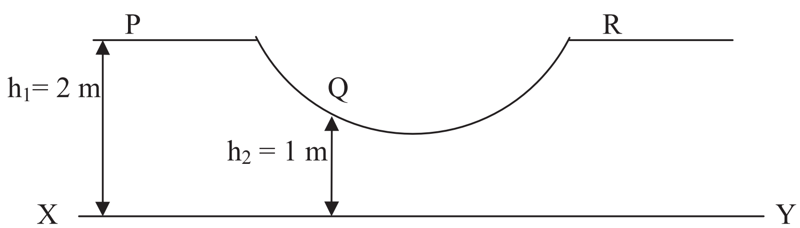
6. Given, m = 51 Kg, t= 10 sec, h= 6 m

The boy does work against his weight (i.e. gravitational force) in climbing. Therefore,

Work done W = mgh = 51 × 9.8 × 6 = 2998.8 Joule

This work is done in 10 sec. Therefore the power output $P = \frac{W}{t} = \frac{2998.8}{10}$ = 299.88 W

7.



Let the speed of particle at Q be v. Obviously, the kinetic energy of particle at P will be zero.

Applying principle of conservation of energy-

Total mechanical energy at P = Total mechanical energy at Q

Kinetic energy at P + Potential energy at P = Kinetic energy at Q + Potential energy at Q

Considering horizontal surface XY as reference level –

$0 + mgh_1 = \frac{1}{2} mv^2 + mgh_2$

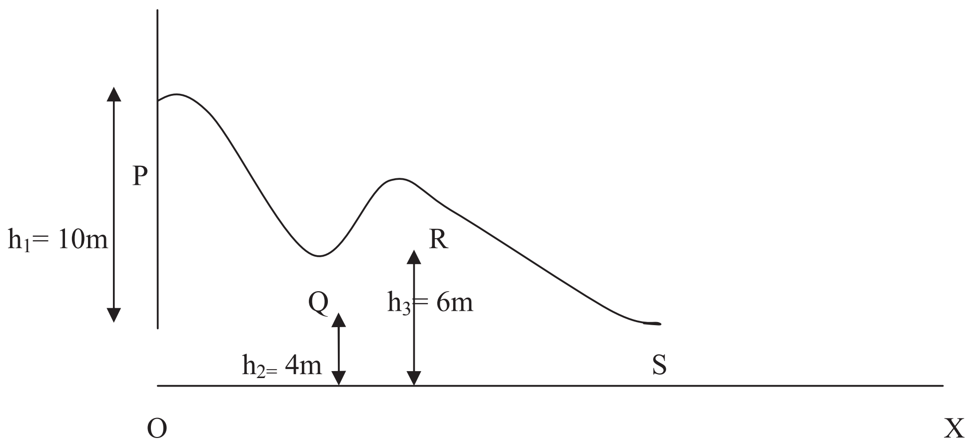
or $gh_1 = \frac{1}{2} v^2 + gh_2$

or $v^2 = 2gh_1 - 2gh_2$

or $v = \sqrt{2gh_1 - 2gh_2} = \sqrt{2g(h_1 - h_2)} = \sqrt{2 \times 10(2-1)}$ = 4.47 m/sec

8.

\



Applying principle of conservation of energy-

Total mechanical energy at P = Total mechanical energy at Q

Kinetic energy at P + Potential energy at P = Kinetic energy at Q + Potential energy at Q

Considering horizontal plane XY as reference level –

$0 + mgh_1 = K_Q + mgh_2$

$K_Q = mgh_1 - mgh_2 = mg(h_1 - h_2)$

$= 2\times 9.8(10 - 4) = 117.6$ Joule

Again using principle of conservation of energy-

Total mechanical energy at P = Total mechanical energy at R

Kinetic energy at P + Potential energy at P = Kinetic energy at R + Potential energy at R

Considering horizontal plane XY as reference level –

$0 + mgh_1 = K_R + mgh_3$

$K_R = mg(h_1 - h_3) = 2 \times 9.8(10 - 6) = 78.4$ Joule

Applying principle of conservation of energy-

Total mechanical energy at P = Total mechanical energy at S

Kinetic energy at P + Potential energy at P = Kinetic energy at S + Potential energy at S

Considering horizontal plane XY as reference level –

$0 + mgh_1 = K_S + 0$

$K_S = mgh_1 = 2 \times 9.8 \times 10 = 196$ Joule

9. By the conservation of angular momentum-

$I_1\omega_1 = I_2\,\omega_2$

Here $I_1 = mr_1^2$, $\omega_1 = \frac{2\pi}{T_1} = \frac{2\pi}{2} = \pi$ radian/sec, $I_2 = m\left(\frac{r_1}{2}\right)^2 = \frac{mr_1^2}{4}$

or $mr_1^2 \times \pi = \frac{mr_1^2}{4} \times \omega_2$

or $\omega_2 = 4\pi$

or $\frac{2\pi}{T_2} = 4\pi$

or $T_2 = 0.5$ sec

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5.16 SUGGESTED READINGS

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UNIT 6: ROTATIONAL MOTION

STRUCTURE:

6.1 INTRODUCTION

In previous units 4 and 5, you have studied some important concepts of Mechanics such as displacement, velocity, acceleration, causes of motion, Newton's laws of motion, linear momentum, work, power, energy etc. You have also studied important conservation principles such as conservation of linear momentum, conservation of energy and conservation of angular momentum. In these units, you have dealt with mainly translatory motion. We have not gone into describing and analyzing the rotational motion of the particles. You should know that rotational (angular) motion also plays an important role in this universe. You have many examples of rotational motion in your life. Rotating galaxies, orbiting planets, bicycle wheels, train wheels, pulleys, door of almirah, ceiling fan in your room etc. have rotational motion involved. Obviously, it is very necessary to study and analyze the rotational motion of particles. Therefore, in this unit we shall study angular velocity, angular acceleration, equations of angular motion, angular momentum and torque. In this unit, we shall also study some important examples and applications based on rotational motion.

6.2 OBJECTIVES

After studying this unit, you should be able to-

- Compute angular velocity and angular acceleration of a particle undergoing rotational motion
- apply equations of angular motion
- relate torque and moment of inertia
- relate linear and angular variables of rotating body
- compute rotational kinetic energy and angular momentum of particles
- solve problems based on rotational motion
- compare linear quantities and angular quantities

6.3 ROTATIONAL MOTION

If an external force applied to a body does not produce any displacement of the particles of the body relative to each other, then the body is called a 'rigid body'. In fact, no real body is perfectly rigid. However, in solid bodies, except rubber etc., the relative displacement by the external force is so small that it can be neglected. Hence usually when we speak of a body, we mean a rigid body.

When a body rotates about a fixed axis, the rotation is known as 'rotatory motion' or 'angular motion' and the axis is known as the 'axis of rotation'. In rotatory motion, every particle of the body moves in a circle and the centres of all these circles lie at the axis of rotation. The rotating blades of an electric fan and the motion of a top are the examples of rotatory motion.

Consider the door of your almirah. When you open the door, the vertical line passing through the hinges is held fixed and that is the axis of rotation. Each particle of the door describes a circle with the centre at the foot of the perpendicular from the particle on the axis. All these circles are horizontal and thus perpendicular to the axis.

Look at the ceiling fan in your room. When it is on, each point on its body goes in a circle. Locate the centres of the circles traced by different particles on the three blades of the fan and the body covering the motor. All these centres lie on a vertical line through the centre of the body. The fan rotates about this vertical line.

6.3.1 Angular Displacement

Let us consider that a particle P is moving in a circle around a point O with a constant speed v. Let X is the initial position of the particle at time t = 0. The instantaneous position of P is expressed by an angle θ between a radial line OP and a reference line OX. P is the position of the particle after time t. θ is the angle subtended at the centre O by particle in time t.

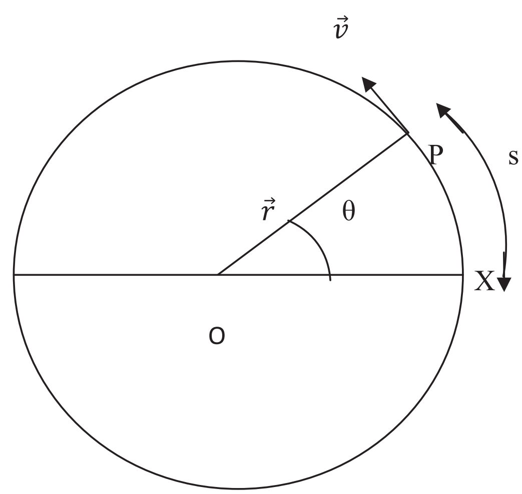

Figure 1

We know, angle = $\frac{\text{arc}}{\text{radius}}$

or $\theta = \frac{XP}{OP}$

or $\theta = \frac{s}{r}$ …..(1)

If a particle starts from position X and P_1 and P_2 be the positions of the particle at time t_1 and t_2 respectively. θ_1 and θ_2 be its angular position at time t_1 and t_2. Suppose in time interval Δt (= $t_2 - t_1$), the particle covers a distance Δs along the circular path. It revolves through the angle $\Delta\theta$ (= $\theta_2 - \theta_1$) during this time interval. The angle of revolution $\Delta\theta$ is called the 'angular displacement'

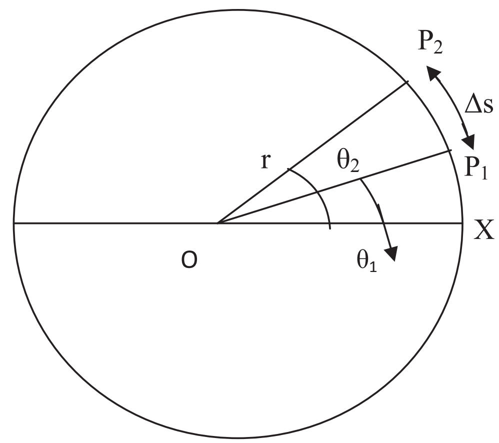

Figure 2

of the particle. If r is the radius of the circle, then the angular displacement is given by-

$\Delta\theta = \frac{\Delta s}{r}$ …..(2)

The unit of the angle or angular displacement is radian.

If $\Delta s = r$, then $\Delta\theta = \frac{1}{1} = 1$ radian

i.e. if the length of the arc of a circle is equal to the radius of the circle, then the angle subtended by the arc at the centre of the circle is 1 radian.

The whole circumference of the circle subtends an angle of 360^0 at the centre of the circle. According to the definition of radian, the angle subtended by the whole circumference ($2\pi r$) at the centre = $\frac{2\pi r}{r} = 2\pi$ radian. Hence 2π radian = 360^0.

6.3.2 Angular Velocity

"The time-rate of change of angular displacement is called angular velocity". It is denoted by Greek letter ω (omega) i.e.

$\omega = \frac{\theta}{t}$

or $\omega = \frac{\Delta\theta}{\Delta t}$ …..(3)

where $\Delta\theta$ is the angular displacement of the particle in time-interval Δt.

The instantaneous angular velocity of the particle is given by -

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

$$\text{or } \omega = \frac{d\theta}{dt} \qquad \text{.....(4)}$$

The unit of angular velocity is radian/sec.

In one complete revolution, the particle undergoes an angular displacement of 2π radian (or 360^0) and it takes the time T (i.e. time of period), then angular velocity of the particle is given by-

$$\omega = \frac{2\pi}{T} \qquad \text{.....(5)}$$

(since $\Delta\theta = 2\pi$, $\Delta t = T$)

If the particle makes n revolutions in 1 second, then

$$\omega = 2\pi n \qquad \text{.....(6)}$$

(since $\frac{1}{T} = n$, frequency)

The angular velocity is the characteristic of the body as a whole.

6.3.3 Angular Acceleration

If the angular velocity of a rotating body about an axis is changing with time then its motion is 'accelerated rotatory motion'. "The time-rate of change of angular velocity of a body about an axis is called angular acceleration of the body about that axis". It is denoted by α (alpha).

If the angular velocity of a body about an axis changes from ω_1 to ω_2 in time-interval $(t_2 - t_1)$, then the angular acceleration of the body about that axis is –

$$\alpha = \frac{\text{change in angular velocity}}{\text{time-interval}}$$

$$= \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

$$\text{or } \alpha = \frac{\Delta\omega}{\Delta t} \qquad \text{.....(7)}$$

The instantaneous angular acceleration of the particle is given by-

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \qquad \text{.....(8)}$$

or $\alpha = \frac{d}{dt}(\frac{d\theta}{dt})$ (since $\omega = \frac{d\theta}{dt}$)

or $\alpha = \frac{d^2\theta}{dt^2}$ …..(9)

The angular acceleration also is the characteristic of the body as a whole. Its unit is radian/sec^2.

6.3.4 Relation between Angular Velocity and Linear Velocity

We know that $\theta = \frac{s}{r}$

or $s = r\,\theta$ …..(10)

Differentiating with respect to t, we get-

$\frac{ds}{dt} = r\frac{d\theta}{dt}$ (since r = constant)

But $\frac{ds}{dt}$ = v (linear velocity of the particle) and $\frac{d\theta}{dt} = \omega$ (angular velocity)

Therefore, $v = r\,\omega$ …..(11)

This is the relation between the magnitudes of linear velocity of a particle and the angular velocity.

In vector form, $\vec{v} = \vec{\omega} \times \vec{r}$ …..(12)

The direction $\vec{\omega}$ is always along the action of rotation, being upwards for a particle moving anti clockwise (the direction of $\vec{r} \times \vec{v}$), in fig. 2, it is normal to the plane of the paper upwards at the center O.

6.3.5 Relation between Angular Acceleration and Linear Acceleration

We know that $v = r\,\omega$

Differentiating with respect to t, we get-

$\frac{dv}{dt} = r\frac{d\omega}{dt}$ (since r = constant)

But $\frac{dv}{dt} = a_T$ (tangential component of the linear acceleration)

and $\frac{d\omega}{dt} = \alpha$ (angular acceleration of the body as a whole)

Therefore, $a_T = r\,\alpha$ …..(13)

This is the relation between tangential linear acceleration of a particle in the body at a distance r from the axis of rotation and the angular acceleration of the body.

We know that the radial (centripetal) acceleration a_R of a particle moving with velocity v in a circle of radius r is $\frac{v^2}{r}$. This can be expressed in terms of angular velocity ω of the body.

$$a_R = \frac{v^2}{r} = \frac{r^2\omega^2}{r} \quad (\text{since } v = r\,\omega)$$

$$a_R = r\,\omega^2 \qquad \text{…..(14)}$$

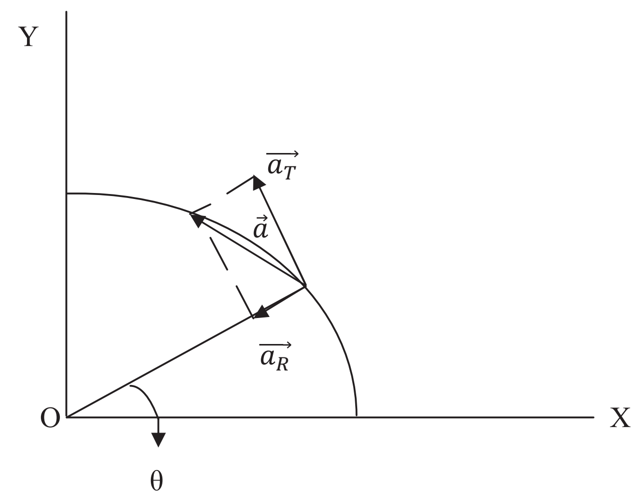

Figure 3

The resultant acceleration $\vec{a}$ of the particle is-

$$\vec{a} = \overrightarrow{a_T} + \overrightarrow{a_R}$$

$$\text{or}\ \ a = \sqrt{a_T^2 + a_R^2}$$

$$= \sqrt{(r\alpha)^2 + (r\omega)^2}$$

$$\text{or}\ \ a = r\sqrt{\alpha^2 + \omega^4} \qquad \text{…..(15)}$$

Example 1: A car is moving with a speed of 20 m/sec on a circular track of radius 400 meter. Its speed is increasing at the rate of 4 m/sec^2. Find out the value of its acceleration.

Solution: The speed of the car moving on a circular track is increasing. Therefore, besides the centripetal (radial) acceleration a_R, the car has a tangential acceleration a_T. a_R and a_T are mutually perpendicular.

Here v = 20 m/sec. r = 400 m, a_T = 4 m/sec^2

Centripetal (radial) acceleration $a_R = \frac{v^2}{r} = \frac{(20)^2}{400}$ = 1 m/sec^2

Therefore, result acceleration a = $\sqrt{a_T^2 + a_R^2}$

$= \sqrt{(4)^2 + (1)^2} = \sqrt{17}$ = 4.12 m/sec^2

Example 2: The moon revolves around the earth in 2.4×10^6 sec in a circular orbit of radius 3.9 $\times 10^5$ Km. Determine the acceleration of the moon towards the earth.

Solution: Given, r = 3.9×10^5 Km = $3.9 \times 10^5 \times 10^3$ m = 3.9×10^8 m, T = 2.4×10^6 sec

Angular velocity of the moon $\omega = \frac{2\pi}{T} = \frac{2\times3.14}{2.4\times10^6}$

$= \frac{6.28\times10^{-6}}{2.4} = 2.62 \times 10^{-6}$ radian/sec

The acceleration of the moon towards the earth, a = r ω^2

$= 3.9 \times 10^8 \times (2.62 \times 10^{-6})^2$

$= 2.68 \times 10^{-3}$ m/sec^2

Self Assessment Question (SAQ) 1: Show that angular acceleration $\vec{\alpha}$ is perpendicular to angular velocity $\vec{\omega}$, if ω is a constant.

Self Assessment Question (SAQ) 2: Calculate the angular speed of a flywheel making 120 revolutions per minute.

Self Assessment Question (SAQ) 3: A particle is revolving round a circular path. What is the direction of acceleration of the particle?

Self Assessment Question (SAQ) 4: Two racing cars of masses m_1 and m_2 are moving in circles of radii r_1 and r_2 respectively. Their speeds are such that each of them makes a complete circle in the same time t. What is the ratio of their angular speed?

6.4 TORQUE

A force is required to produce linear acceleration in a particle. In the similar way, a torque (or moment of force) is required to produce angular acceleration in a particle about an axis.

When an external force acting on a body has a tendency to rotate the body about an axis, then the force is said to exert a 'torque' upon the body about that axis. "The torque (or moment of a force) about an axis of rotation is equal to the product of the magnitude of the force and the perpendicular distance of the line of action of the force from the axis of rotation".

In the figure 4 is shown a body which is free to rotate about an axis passing through a point O and perpendicular to the plane of the paper. Let a force F be applied on the body in the plane of the paper to rotate the body about this axis.

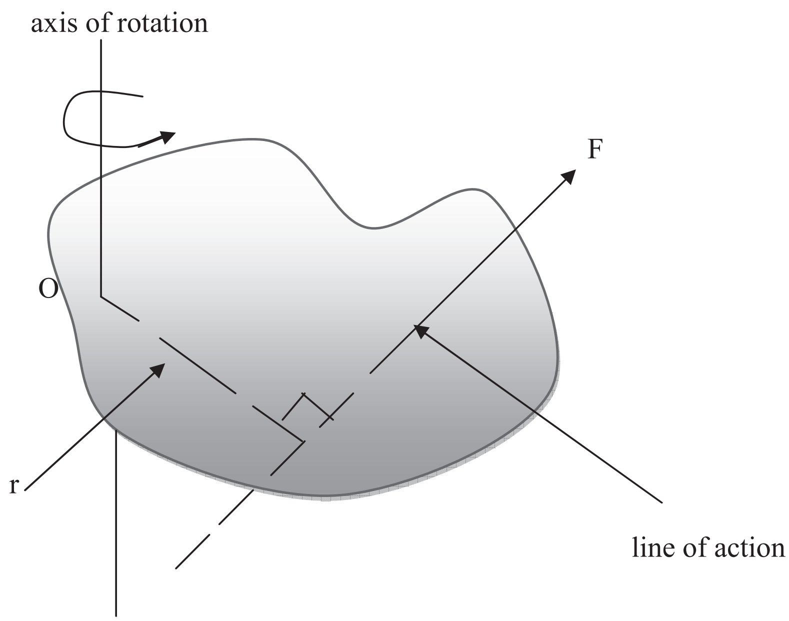

Figure 4

Let r be the perpendicular distance of the line of action of the force from the point O, then torque or moment of force F about the axis of rotation is given as-

$\tau = F \times r$ …..(16)

If the torque tends to rotate the body anticlockwise then it is taken as positive; if clockwise then negative. The unit of torque is Newton-meter and like force, it is a vector quantity.

In vector form, $\vec{\tau} = \vec{r} \times \vec{F}$ …..(17)

where $\vec{r}$ is the position vector of the point at which force acts with respect to the reference point.

Its scalar magnitude $\tau = r F \sin\theta$ .....(18)

If $\theta = 90^0$ i.e. r is perpendicular to the line of action of force, then

$\tau = r F \sin 90^0$

$= r F$ (maximum)

i.e. the torque is maximum when the force is applied at the right angle to $\vec{r}$. This is why in opening or closing a heavy revolving door the force is applied (by hand) at right angles to the door at its outer edge. Besides this, the torque depends also on the position of the point relative to origin at which the force is applied. If force is applied at the origin (i.e. $\vec{r}$ is zero) then no torque is produced. In this situation, the body will not rotate how-so-ever large the force may be. This is why we cannot open or close a door by applying force at the hinge.

On the contrary, greater is the distance of the line of action from origin, larger is the moment of force or torque about O; or smaller the force required to rotate the body. This fact is used in daily life. The handle revolving the grinding machine is fixed quite far from the pivot, the water-pump is fitted with a long handle and the handle of a door is fixed at a large distance from the pivot. The handle of a screw-driver is made wide because of the same reason.

6.5 MOMENT OF INERTIA

A body rotating about an axis resists any change in its rotational motion (angular velocity). On account of this property the body is said to possess a 'moment of inertia' or 'rotational inertia' about that axis. "The property of a body by virtue of which it opposes any change in its state of rotation about an axis is called the 'moment of inertia' of the body about that axis". It is denoted by I.

"The moment of inertia of a particle about an axis is given by the product of the mass of the particle and the square of the distance of the particle from the axis of rotation".

If a particle of mass m is at a distance r from an axis of rotation, its moment of inertia I about that axis is given as-

$I = mr^2$ .....(19)

Let us consider a rigid body of mass M. We have to find out the moment of inertia about a vertical axis passing through O (Figure 5). If m_1, m_2, m_3, ...be the masses of the particles composing the body and r_1, r_2, r_3, ...their respective distances from the axis of rotation, the moment of inertia I of the body about that axis is equal to the sum of the moments of inertia of all the particles i.e.

$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots\ldots\ldots$ .....(20)

or $I = \Sigma mr^2$ .....(21)

Here Σ (sigma) means the sum of all terms.

Figure 5

For a body having a continuous distribution of matter, $I = \int r^2\, dm$ .....(22)

where dm is the mass of an infinitesimally small element of the body taken at a distance r from the axis of rotation. Hence, "the moment of inertia of a rigid body about a given axis is the sum of the products of the masses of its particles by the square of their respective distances from the axis of rotation".

The unit of moment of inertia is Kg-m^2.

Obviously, the moment of inertia of a body about an axis depends not only on the mass of the body but also upon the manner in which the mass is distributed around the axis of rotation.

6.5.1 Radius of Gyration

The radius of gyration of a body about an axis of rotation is defined as the distance of a point from the axis of rotation at which whole mass of the body were assumed to be concentrated, its moment of inertia about the given axis would be the same as with its actual distribution of mass. It is usually denoted by the letter k.

If M is the total mass of the body; its moment of inertia in terms of its radius of gyration k can be written as-

$I = M k^2$ …..(23)

or $k = \sqrt{\frac{I}{M}}$ …..(24)

Thus "the square root of the ratio of moment of inertia of the body about the given axis of rotation to its mass is called radius of gyration of the body about the given axis".

6.5.2 Physical significance of Moment of Inertia

According to Newton's first law of motion, we know that if a body is at rest or moving with a uniform speed along a straight line, then an external force is necessary to change its state. This property of bodies is called 'inertia'. Greater the mass of a body, greater is the force required to bring a change in its position of rest or in its linear velocity (i.e. to produce linear acceleration in it). In this way, the mass of a body is a measure of its inertia.

Similarly, in order to rotate a body (initially at rest) about an axis or to change the angular velocity of a rotating body (i.e. to produce an angular acceleration in it), a torque has to be applied on the body. This is described by saying that the body has a 'moment of inertia' about the axis of rotation. The greater the moment of inertia of a body about an axis, the greater is the torque required to rotate, or to stop, the body about that axis. Thus, the moment of inertia plays the same role in the rotational motion as mass plays in translational motion.

There is a difference between inertia and moment of inertia of a body. The inertia of a body depends only upon the mass of the body. But the moment of inertia of a body about an axis depends not only on the mass of the body but also upon the distribution of its mass about the axis of rotation.

6.5.3 Practical applications of Moment of Inertia

An important use of the property of moment of inertia is made in stationary engines. The torque rotating the shaft of an engine changes periodically and so the shaft cannot rotate uniformly. To keep its rotation uniform, a large heavy wheel is attached with the shaft. This wheel is called 'flywheel' and it has a large moment of inertia. As the shaft rotates, the flywheel also rotates. Due to its large moment of inertia, the flywheel (and hence the shaft) continues to rotate almost uniformly in spite of the changing torque. A small flywheel is attached to the bottom of toy-motor. The flywheel is rotated by rubbing it on the ground and the motor is left for running. Due to the moment of inertia of the flywheel, the motor continues moving for some time.

The moment of inertia plays vital role in our daily life. In cycle, rickshaw, bullock-cart, etc., the moment of inertia of the wheels is increased by concentrating most of the mass at the rim of the wheel and connecting the rim to the axle of the wheel through spokes. It is due to the large moment of inertia of the wheels that when we stop cycling, the wheels of the cycle continue rotating for some time.

6.5.4 Moment of Inertia of certain regular bodies

(i) Thin Rod: If the mass of a thin rod is M and its length is L, then the moment of inertia of the rod about an axis passing through its centre of gravity and perpendicular to its length is given by-

$$I = \frac{ML^2}{12}$$

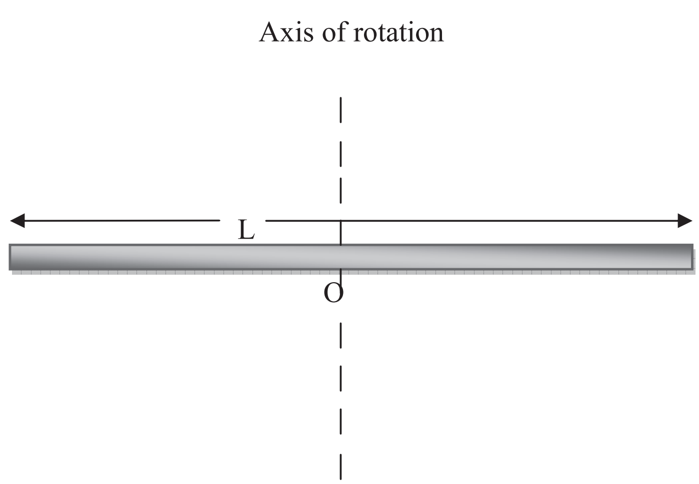

Figure 6

(ii) Rectangular Plate: The moment of inertia of a plate of mass M, length L and breadth B about an axis passing through its centre of gravity and perpendicular to its plane is given by-

$$I = M\left(\frac{L^2 + B^2}{12}\right)$$

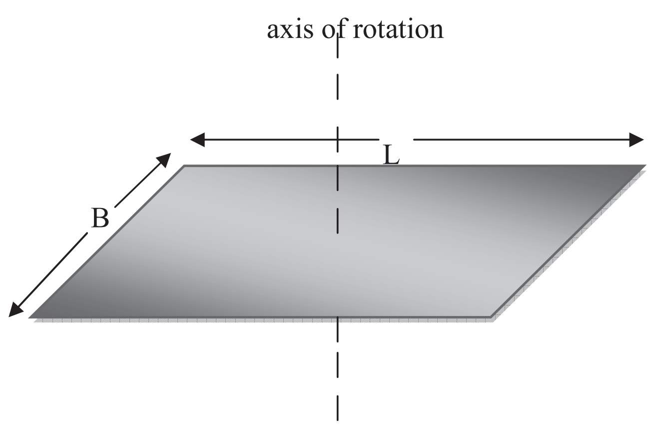

Figure 7

(iii) Ring: If the mass of a ring is M and its radius is R then its moment of inertia about its own geometrical axis is given by-

$I = MR^2$

(iv) Solid Cylindrical Rod: The moment of inertia of a solid cylinder of mass M, length L and radius R about an axis passing through its centre of gravity and perpendicular to its length is given by-

$I = M\left(\frac{L^2}{12} + \frac{R^2}{4}\right)$

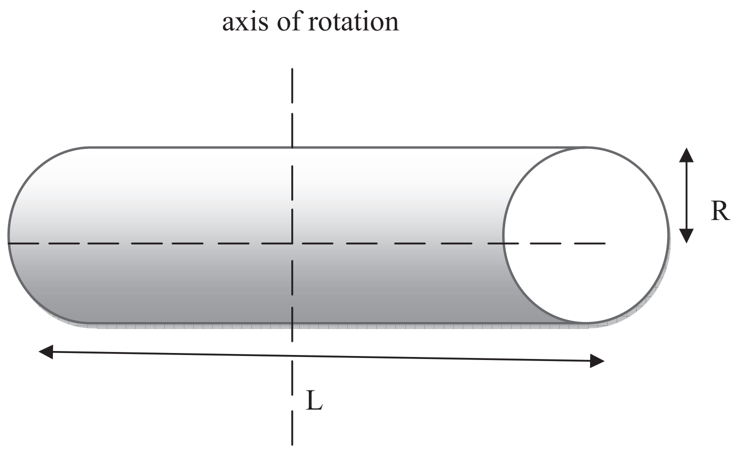

Figure 8

(v) Solid Disc: If the mass of a disc is M and the radius is R, then its moment of inertia about an axis passing through its centre of gravity and perpendicular to its plane is given by-
$I = \frac{1}{2} M R^2$

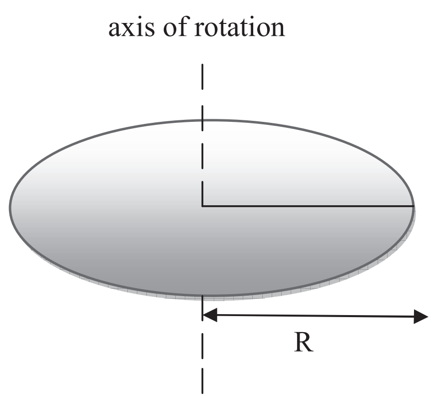

Figure 9

(vi) Solid Sphere: If the mass of a solid sphere is M and its radius is R , then its moment of inertia about a diameter is given by-

$I = \frac{2}{5} M R^2$

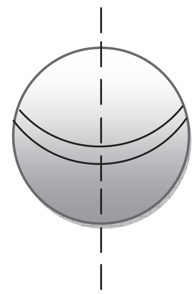

Figure 10

(vii) Spherical Shell: If the mass of a spherical shell is M and its radius is R, then its moment of inertia about a diameter is given by-

$I = \frac{2}{3} M R^2$

6.6 RELATION BETWEEN TORQUE AND MOMENT OF INERTIA

Let us consider a body acted upon by a torque τ, it is rotating about an axis passing through a fixed point O. Suppose it has a constant angular acceleration α. The angular acceleration of all the particles of the body will be the same (i.e. α), but their linear accelerations will be different. Suppose, the mass of one particle of the body is m_1 and its distance from the axis of rotation is r_1. Then the linear acceleration of this particle is given by-

$a_1 = r_1 \alpha$; (as $v_1 = \omega r_1$; $\frac{dv_1}{dt} = r_1 \frac{d\omega}{dt}$)

If F_1 be the force acting on this particle, then

F_1 = mass × acceleration = $m_1 a_1$

$= m_1 r_1 \alpha$

The moment of this force about the axis of rotation passing through O = $F_1 \times r_1$

$= m_1 r_1 \alpha \times r_1 = m_1 r_1^2 \alpha$

Similarly, if the masses of other particles be m_2, m_3, ……. and their respective distances from the axis of rotation be r_2, r_3, ………, then the torques acting on them will be $m_2r_2^2\alpha$, $m_3r_3^2\alpha$, …… respectively. The torque τ acting on the whole body will be the sum of the torques acting on all the particles i.e.

$$\tau = m_1r_1^2\alpha + m_2r_2^2\alpha + m_3r_3^2\alpha + \ldots\ldots\ldots\ldots$$

$$= (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots\ldots\ldots)\alpha = (\Sigma\, mr^2)\,\alpha$$

But $\Sigma\, mr^2$ is the moment of inertia I of the body about the axis of rotation. Hence

$$\tau = I \times \alpha$$

torque = moment of inertia × angular acceleration

If $\alpha = 1$, then $\tau = I$ i.e. the moment of inertia of a body about an axis is equal to the torque required to produce unit angular acceleration in the body about that axis.

Example 3: A flywheel of mass 20 Kg and radius of gyration 100 cm is being acted on by a torque of 20 N-m. Determine the angular acceleration produced.

Solution: Given M = 20Kg, k = 100 cm = 1 m, τ = 20 N-m

We know $I = M\,k^2$

$$= 20 \times (1)^2 = 20 \text{ Kg m}^2$$

Again using $\tau = I \times \alpha$

or $\alpha = \frac{\tau}{I} = \frac{20}{20} = 1$ radian/sec^2

Example 4: The torque τ acting on a body of moment of inertia I about the axis of rotation is given by $\tau = (at^2 + bt + c)\, I$, where a, b, c are constants. Express angular displacement of the body (starting from rest at t = 0) as a function of t.

Solution: We know the relation between torque and moment of inertia as-

$\tau = I \times \alpha$ …..(i)

Given $\tau = (at^2 + bt + c)\, I$ …..(ii)

Comparing equations (i) and (ii), we get-

$$\alpha = at^2 + bt + c$$

or $\frac{d^2\theta}{dt^2} = at^2 + bt + c$

Integrating both sides-

$$\int \frac{d^2\theta}{dt^2}\, dt = \int (at^2 + bt + c)dt$$

or $\frac{d\theta}{dt} = \frac{at^3}{3} + \frac{bt^2}{2} + ct + A$ …..(iii)

where A is constant of integration.

At t = 0, $\frac{d\theta}{dt} = 0$

Therefore, 0 = 0 + A or A = 0

Putting for A in equation (iii) we get-

$\frac{d\theta}{dt} = \frac{at^3}{3} + \frac{bt^2}{2} + ct$ …..(iv)

Again integrating with respect to t,

$$\int \frac{d\theta}{dt}\, dt = \int \left(\frac{at^3}{3} + \frac{bt^2}{2} + ct\right) dt$$

or $\theta = \frac{at^4}{12} + \frac{bt^3}{6} + \frac{ct^2}{2} + B$ …..(v)

where B is constant of integration.

At t = 0, θ = 0, therefore from equation (v), we have-

B = 0

Therefore $\theta = \frac{at^4}{12} + \frac{bt^3}{6} + \frac{ct^2}{2}$

Self Assessment Question (SAQ) 5: Two circular discs P and Q of same mass and same thickness are made of two different metals whose densities are d_P and d_Q ($d_P > d_Q$). Their moments of inertia about the axes passing through their centres of gravity and perpendicular to their planes are I_P and I_Q. Which one has greater moment of inertia.

Self Assessment Question (SAQ) 6: A torque of 4×10^{-4} N-m is to be applied to produce an angular acceleration of 8 radian/sec^2 in a flywheel. Estimate the moment of inertia of the flywheel.

Self Assessment Question (SAQ) 7: Why is a ladder more likely to slip when you are high up on it than when you just begin to climb?

Self Assessment Question (SAQ) 8: Why is it more difficult to revolve a stone by tieing it to a longer string than by tieing it to a shorter string?

Self Assessment Question (SAQ) 9: Choose the correct option-

(i) In rotating motion, moment of inertia-

(a) imparts angular acceleration (b) imparts angular deceleration (c) aids change in rotational motion (d) opposes the change in rotational motion

(ii) In rotatory motion the physical quantity that imparts angular acceleration or deceleration is-

(a) moment of inertia (b) torque (c) force (d) angular velocity

(iii) Moment of inertia in rotational motion has its analogue in translatory motion-

(a) mass (b) torque (c) force (d) displacement

6.7 EQUATIONS OF ANGULAR MOTION

When a body rotates with constant angular acceleration, then the relations among its angular velocity, angular displacement, angular acceleration and time can be expressed by simple equations as in the case of translatory motion.

First Equation: If ω_0 be the initial angular velocity of a body rotating about a fixed axis with constant angular acceleration α, then its angular velocity after time t is given by-

$\omega = \omega_0 + \alpha\, t$

Proof: We know that angular acceleration $\alpha = \frac{d\omega}{dt}$

or $d\omega = \alpha\, dt$

Integrating both sides, we get-

$\int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha\, dt$

or $(\omega - \omega_0) = \alpha\,(t-0)$

or $\omega = \omega_0 + \alpha\, t$ …..(25)

Second Equation: If ω_0 be the initial angular velocity of a body rotating about a fixed axis with constant angular acceleration α, then the angle traced by the body after time t is –

$\theta = \omega_0\, t + \frac{1}{2}\alpha\, t^2$

Proof: We know that angular velocity $\omega = \frac{d\theta}{dt}$

But $\omega = \omega_0 + \alpha\, t$

Therefore, $\omega_0 + \alpha\, t = \frac{d\theta}{dt}$

or $\frac{d\theta}{dt} = \omega_0 + \alpha\, t$

$d\theta = (\omega_0 + \alpha\, t)\, dt$

Integrating both sides-

$\int_0^{\theta} d\theta = \int_0^{t} (\omega_0 + \alpha\, t)\, dt$

or $\theta = \omega_0\, (t-0) + \alpha \left(\frac{t^2}{2} - 0\right)$

or $\theta = \omega_0\, t + \frac{1}{2}\alpha\, t^2$ …..(26)

Third Equation: If ω_0 be the initial angular velocity of a body rotating about a fixed axis with constant angular acceleration α and the angular displacement in time t be θ then its angular velocity after time t is given by-

$\omega^2 = \omega_0^2 + 2\,\alpha\,\theta$

Proof: By first equation, $\omega = \omega_0 + \alpha\, t$

Squaring both sides-

$\omega^2 = (\omega_0 + \alpha\, t)^2$

or $\omega^2 = \omega_0^2 + \alpha^2 t^2 + 2\,\omega_0 \alpha t$

or $\omega^2 = \omega_0^2 + 2\alpha\,(\omega_0 t + \frac{1}{2}\alpha\, t^2)$

or $\omega^2 = \omega_0^2 + 2\alpha\theta$ …..(27)

(using equation 26)

6.7.1 Linear and Angular variables of a rotating body

The following table represents the variables or quantities in linear and angular motion-

Linear Motion	**Angular Motion**
Linear displacement s	Angular displacement θ
Linear velocity $v = \frac{ds}{dt}$	Angular velocity $\omega = \frac{d\theta}{dt}$
Linear acceleration $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$	Angular acceleration $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
Mass m	Moment of inertia I
Force F	Torque τ
$F = ma$	$\tau = I\ \alpha$
$W = Fs$	$W = \tau\ \theta$
$p = mv$	$J = I\ \omega$
$K = \frac{1}{2}\ mv^2$	$K = \frac{1}{2}\ I\ \omega^2$
$v = u + at$	$\omega = \omega_0 + \alpha\ t$
$s = ut + \frac{1}{2}\ at^2$	$\theta = \omega_0\ t + \frac{1}{2}\ \alpha\ t^2$
$v^2 = u^2 + 2\ as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$

Example 5: The wheel of a car is completing 1200 rotations in 1 minute. On pressing the accelerator of the car, the wheel makes 2400 rotations in 1 minute. Compute its angular acceleration and the angular displacement in 10 sec.

Solution: Here $\omega_0 = 2\pi n_0 = 2\pi \times \frac{1200}{60} = 40\ \pi$ radian/sec, $\omega = 2\pi n = 2\pi \times \frac{2400}{60} = 80\ \pi$ radian/sec, t = 10 sec

Using $\omega = \omega_0 + \alpha\ t$

$$80\ \pi = 40\ \pi + \alpha \times 10$$

or $\alpha = 4\ \pi$ radian/sec^2

Now using $\omega^2 = \omega_0^2 + 2\alpha\theta$

$$(80\ \ )^2 = (40\ \pi\ )^2 + 2 \times 4\ \pi\ \theta$$

or $\theta = 600\ \pi$ radian

Example 6: Moment of inertia of a ring is 3 Kg-m^2. It is rotated for 20 sec from its rest position by a torque of 6 N-m. Calculate the work done.

Solution: I = 3 Kg-m^2, $\omega_0 = 0$, t = 20 sec, $\tau = 6$ N-m

Using $\tau = I\ \alpha$

or $\alpha = \frac{\tau}{I} = \frac{6}{3} = 2$ radian/sec^2

Using $\theta = \omega_0 t + \frac{1}{2} \alpha t^2$

$$\theta = 0\ (20) + \frac{1}{2}\ (2)\ (20)^2 = 400 \text{ radians}$$

Work done $W = \tau\ \theta$

$$= 6 \times 400 = 2400 \text{ Joule}$$

Self Assessment Question (SAQ) 10: A motor of an engine is rotating about its axis with an angular velocity of 100 revolutions/min. It comes to rest in 15 sec, after being switched off. Assuming constant angular deceleration, calculate the number of revolutions made by it before coming to rest.

6.8 ROTATIONAL KINETIC ENERGY

Let us consider a rigid body rotating about an axis with a uniform angular velocity ω. Each particle in the body has a certain kinetic energy. The angular velocity of each particle of the body will be same equal to ω but their linear velocities will be different. Suppose a particle of the body whose mass is m_1, is at a distance r_1 from the axis of rotation. Let v_1 be the linear velocity of this particle. Therefore,

$v_1 = r_1\ \omega$

Kinetic energy of this particle $K_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 (r_1\ \omega)^2 = \frac{1}{2} m_1\ r_1^2\ \omega^2$

Similarly, kinetic energies of other particles of the body having masses m_2, m_3, ………and distances r_2, r_3, ………from the axis of rotation. Then kinetic energies of these particles are given as-

$K_2 = \frac{1}{2} m_2\ r_2^2\ \omega^2$, $K_3 = \frac{1}{2} m_3\ r_3^2\ \omega^2$, ……………

The kinetic energy of the whole body will be equal to the sum of the kinetic energies of all the particles. Therefore, Kinetic energy of whole body $K = K_1 + K_2 + K_3 + \ldots$…………..

or $K = \frac{1}{2} m_1\ r_1^2\ \omega^2 + \frac{1}{2} m_2\ r_2^2\ \omega^2 + \frac{1}{2} m_3\ r_3^2\ \omega^2 + \ldots$

$$= \frac{1}{2} (m_1\ r_1^2 + m_2\ r_2^2 + m_3\ r_3^2 + \ldots)\ \omega^2$$

$$= \frac{1}{2} (\Sigma\ mr^2)\ \omega^2$$

Since $\Sigma\ mr^2 = I$, moment of inertia

$K = \frac{1}{2} I \omega^2$ …..(28)

This is the expression for rotational kinetic energy of a body. Obviously kinetic energy of rotation is equal to half the product of moment of inertia of the body and the square of the angular velocity of the body.

From above expression, $I = \frac{2K}{\omega^2}$

If $\omega = 1$, then $I = 2K$

i.e. the moment of inertia of a body rotating about an axis with unit angular velocity equals twice the kinetic energy of rotation about that axis.

If a body rotating about an axis is simultaneously moving along a straight line, then its total kinetic energy will be ($\frac{1}{2} mv^2 + \frac{1}{2} I \omega^2$), where v is the linear velocity of the body.

6.9 ANGULAR MOMENTUM

"The moment of linear momentum of a particle rotating about an axis is called angular momentum of the particle". It is denoted by J.

If a particle be rotating about an axis of rotation, then J = linear momentum × distance

$= p \times r$

$= mv \times r$ (since $p = mv$)

or $J = mvr$

where m, v and r are the mass of the particle, linear velocity and distance of particle from axis of rotation respectively.

But $v = r\omega$, where ω is the angular velocity

Therefore, $J = m (r\omega) r = mr^2 \omega$

or $J = I \omega$ …..(29)

where $mr^2 = I$ = Moment of Inertia of particle about the axis of rotation

Let us suppose a body be rotating about an axis with an angular velocity ω. All the particles of the body will have the same angular velocity ω but different linear velocities.

Let a particle be at a distance r_1 from the axis of rotation, the linear velocity of this particle is given by-

$v_1 = r_1\omega$

If m_1 be the mass of the particle, then its linear momentum $p_1 = m_1v_1$

The moment of this momentum about the axis of rotation i.e. angular momentum of the particle J_1 = linear momentum × distance

$= p_1 \times r_1$

$= m_1v_1 \times r_1$

$= m_1(r_1\omega) \times r_1$

$= m_1r_1^2\ \omega$

Similarly, if the masses of other particles be m_2, m_3, …… and their respective distances from the axis of rotation be r_2, r_3, ……… , then the moments of their linear momenta about the axis of rotation will be $m_2r_2^2\ \omega$, $m_3r_3^2\ \omega$, ………..respectively. The sum of these moments of linear momenta of all the particles i.e. the angular momentum of the body is given by-

$J = m_1r_1^2\ \omega + m_2r_2^2\ \omega + m_3r_3^2\ \omega + \ldots\ldots\ldots$

$= (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots\ldots)\omega$

$= (\sum mr^2)\omega$

or $J = (\sum mr^2)\omega$ …..(30)

But $(\sum mr^2) = I$, the moment of inertia of the body about the axis of rotation

Therefore, angular momentum $J = I\ \omega$ …..(31)

The unit of angular momentum is Kg-m^2/sec. It is a vector quantity.

In vector form, $\vec{J} = \vec{r} \times \vec{p}$ …..(32)

or $\vec{J} = r\ p\ \sin\theta\ \hat{n}$ …..(33)

where θ is the angle between $\vec{r}$ and $\vec{p}$ and n^ is the unit vector perpendicular to the plane containing $\vec{r}$ and $\vec{p}$.

Magnitude of angular momentum $J = r\ p\ \sin\theta$ …..(34)

6.9.1 Relation between Torque and Angular Momentum

We know that relation for angular momentum-

$J = I\ \omega$ …..(35)

The rate of change of angular momentum-

$$\frac{\Delta J}{\Delta t} = I\,\frac{\Delta\omega}{\Delta t}$$

$$= I\,\alpha \qquad \text{…..(36)}$$

(since $\frac{\Delta\omega}{\Delta t} = \alpha$, angular acceleration)

But $I\,\alpha = \tau$ (Torque)

Therefore, $\frac{\Delta J}{\Delta t} = \tau$ …..(37)

i.e. the time-rate of change of angular momentum of a body is equal to the external torque acting upon the body. The equation (37) represents the relation between torque and angular momentum. This formula is similar to the formula $\frac{\Delta p}{\Delta t} = F$, for linear motion.

In vector form-

$$\vec{J} = \vec{r} \times \vec{p}$$

Differentiating both sides with respect to time t, we get-

$$\frac{d\vec{J}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$= \left(\frac{d\vec{r}}{dt} \times \vec{p}\right) + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right)$$

$$= (\vec{v} \times m\vec{v}) + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right) \qquad \left(\text{since } \frac{d\vec{r}}{dt} = \vec{v} \text{ and } \vec{p} = m\vec{v}\right)$$

$$= m(\vec{v} \times \vec{v}) + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right)$$

$$= 0 + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right) \qquad (\text{since } \vec{v} \times \vec{v} = 0)$$

or $\frac{d\vec{J}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$ …..(38)

By Newton's second law, $\frac{d\vec{p}}{dt} = \vec{F}$

Therefore, $\frac{d\vec{J}}{dt} = \vec{r} \times \vec{F}$ …..(39)

But $\vec{r} \times \vec{F} = \vec{\tau}$, the torque acting on the particle

Therefore, equation (39) becomes-

$\frac{d\vec{J}}{dt} = \vec{\tau}$ …..(40)

i.e. the time-rate of change of angular momentum of a particle is equal to the torque acting on the particle.

6.9.2 Relation between Angular Momentum and Rotational Kinetic Energy

We know that rotational kinetic energy $K = \frac{1}{2} I \omega^2$

Multiplying and dividing by I in right hand side of the above expression, we get-

$K = \frac{1}{2} \frac{I^2\omega^2}{I}$

$= \frac{1}{2} \frac{(I\omega)^2}{I} = \frac{1}{2} \frac{J^2}{I}$ (since $I\omega = J$)

Therefore, $K = \frac{J^2}{2I}$ …..(41)

This is the relation between angular momentum and rotational kinetic energy.

Example 7: A body of mass 1 Kg is rotating on a circular path of diameter 2 m at the rate of 10 rotations in 31.4 sec. Calculate the angular momentum and rotational kinetic energy of the body.

Solution: Given, m = 1 Kg, r = 2 m/2 = 1 m, n = 10/31.4 rotations/sec

Angular velocity $\omega = 2\pi n = 2 \times 3.14 \times 10/31.4 = 2$ radian/sec

Moment of inertia $I = mr^2 = 1\,(1)^2 = 1$ Kg-m^2

Angular momentum $J = I\omega = 1 \times 2 = 2$ Kg-m^2/sec

Rotational kinetic energy $K = \frac{1}{2} I \omega^2 = \frac{1}{2} \times 1 \times (2)^2 = 2$ Joule

Self Assessment Question (SAQ) 11: A solid sphere is rolling on a table. What fraction of its total kinetic energy is rotational?

6.10 SUMMARY

In the present unit, we have studied about rotational motion, torque, moment of inertia and rotational kinetic energy of a body. We have studied about different rotational variables like angular displacement, angular velocity, angular acceleration etc. Angular velocity is defined as the time-rate of change of angular displacement while the time-rate of change of angular velocity is called as angular acceleration. We have also established the relationships between angular velocity and linear velocity as $v = r\omega$ and between angular acceleration and linear acceleration as

$a = r\alpha$. In the unit, we have studied about torque and its importance with examples. The torque or moment of force is given as the product of force applied and the perpendicular of line of action of force from the axis of rotation. In this unit, we have also covered moment of inertia and its physical significance with some practical applications. We have derived an important expression which relates moment of inertia and torque as $\tau = I\,\alpha$. Three important equations of angular (rotatory) motion have been derived in the unit. We have highlighted linear and angular variables of a rotating body. We have also established the expression of rotational kinetic energy of a body as $K = \frac{1}{2} I\,\omega^2$ and defined moment of inertia in terms of rotational kinetic energy as the twice of the kinetic energy of rotation about axis of rotation if the body is rotating with unit angular velocity. In the unit, we have also covered angular momentum, relation between torque & angular momentum and relation between angular momentum & rotational kinetic energy. We have included examples and self assessment questions (SAQs) to check your progress.

6.11 GLOSSARY

Rotational- the action of moving in a circle

Rigid- not able to be changed

Relative- considered in relation or in proportion to something else

Angular- having angles

Characteristic- a quality typical of a thing

Radial- arranged in lines coming out from a central point to the edge of a circle

Situation- a set of circumstances existing at a particular time and in a particular place

Resist- to oppose, withstand the action or effect of

Manner- a way in which something is done or happens

Distribution- the action of distributing something

Necessary- needing to be done or present, essential

Variable- often changing or likely to change , not consistent

6.12 TERMINAL QUESTIONS

1. Define angular velocity and write its unit. Give its relation with linear velocity. Show that $\vec{\omega} = (\vec{r} \times \vec{v} / r^2)$.