REAL ANALYSIS

(MTS6 B10)

VI SEMESTER

CORE COURSE

B.Sc. MATHEMATICS

(2019 Admission onwards)

CBCSS

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SCHOOL OF DISTANCE EDUCATION

Self Learning Material

B.Sc. Mathematics (Sixth Semester)

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MTS6 B10 : REAL ANALYSIS

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Chapter 1

Introduction[1]

1.1 The Real Line

A convenient and familiar geometric interpretation of the real number system is the real line. In this interpretation, the absolute value $|a|$ of an element a in $\mathbb{R}$ is regarded as the distance from a to the origin 0. More generally, the distance between elements a and b in $\mathbb{R}$ is $|a-b|$.

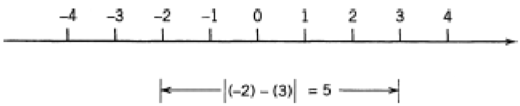

Figure 1.1: The distance between $a = -2$ and $b = 3$

Definition 1. Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Then the ε **-neighborhood** of a is the set $V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\}$

[1] Introductory chapter. Topics mentioned in the Syllabus starts from Page 19

Figure 1.2: An $\varepsilon-$neighborhood of a

Theorem 1.1.1. *Let $a \in \mathbb{R}$. If x belongs to the neighborhood $V_\varepsilon(a)$ for every $\varepsilon > 0$, then $x = a$.*

Proof. Let $a \in \mathbb{R}$ and x belongs to the neighborhood $V_\varepsilon(a)$ for every $\varepsilon > 0$.
That is, $|x - a| < \varepsilon$ for every $\varepsilon > 0$. Suppose to the contrary that $|x - a| > 0$. Then if we take $\varepsilon_0 = \frac{|x-a|}{2}$, we have $0 < \varepsilon_0 < |x - a|$. Therefore it is false that $|x - a| < \varepsilon$ for every $\varepsilon > 0$ and we conclude that $|x - a| = 0$. Which implies that $x = a$. $\square$

Theorem 1.1.2. Archimedean Property *If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x \leq n_x$.*

We now know that there exists at least one irrational real number, namely $\sqrt{2}$. Actually there are "more" irrational numbers than rational numbers in the sense that the set of rational numbers is countable, while the set of irrational numbers is uncountable. We next show that the set of rational numbers is "dense" in $\mathbb{R}$ in the sense that given any two real numbers there is a rational number between them.

Theorem 1.1.3. The Density Theorem *If x and y are any real numbers with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.*

Corollary 1.1.1. *If x and y are real numbers with $x < y$, then there exists an irrational number z such that $x < z < y$.*

Theorem 1.1.4. Characterization Theorem *If S is a subset of $\mathbb{R}$ that contains at least two points and has the property*

$$if\ x, y \in S\ and\ x < y,\ then\ [x, y] \subset S, \tag{1.1}$$

then S is an interval.

1.2 Sequences

Definition 2. A **sequence of real numbers** (or a sequence in $\mathbb{R}$) is a function defined on the set $\mathbb{N} = \{1, 2, ...\}$ of natural numbers whose range is contained in the set $\mathbb{R}$ of real numbers.

Definition 3. A sequence $X = (x_n)$ in $\mathbb{R}$ is said to **converge** to $x \in \mathbb{R}$, or x is said to be a **limit** of (x_n), if for every $\varepsilon > 0$ there exists a natural number $K(\varepsilon)$ such that for all $n \geq K(\varepsilon)$, the terms x_n satisfy $|x_n - x| < \varepsilon$.
If a sequence has a limit, we say that the sequence is **convergent**; if it has no limit, we say that the sequence is **divergent**.

Definition 4. A sequence $X = (x_n)$ of real numbers is said to be **bounded** if there exists a real number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem 1.2.1. *A convergent sequence of real numbers is bounded.*

Theorem 1.2.2.

(a) Let $X = (x_n)$ and $Y = (y_n)$ be sequences of real numbers that converge to x and y, respectively, and let $c \in \mathbb{R}$. Then the sequences

$X+Y$, X–Y, $X.Y$, and cX converge to $x+y$, x–y, xy, and cx, respectively.

(b) If $X=(x_n)$ converges to x and $Z=(z_n)$ is a sequence of nonzero real numbers that converges to z and if $z \neq 0$, then the quotient sequence X/Z converges to x/z.

Theorem 1.2.3. *If $X=(x_n)$ is a convergent sequence and if $a \leq x_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq \lim(x_n) \leq b$.*

Theorem 1.2.4. Squeeze Theorem *Suppose that $X=(X_n), Y=(Y_n)$, and $Z=(z_n)$ are sequences of realnumbers such that,*

$$x_n \leq y_n \leq z_n \qquad for\ all\ n \in (N),$$

and that $\lim(x_n)=\lim(z_n)$. Then $Y=(Y_n)$ is convergent and $\lim(x_n)=\lim(y_n)=\lim(z_n)$

Definition 5. Let $X=(x_n)$ be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a strictly increasing sequence of natural numbers. Then the sequence $X'=(x_{n_k})$ given by

$$(x_{n_1}, x_{n_2}, x_{n_3}, \cdots, x_{n_k}, \cdots)$$

is called a **subsequence** of X.

Theorem 1.2.5. The Bolzano – Weierstrass Theorem *A bounded sequence of real numbers has a convergent subsequence.*

Definition 6. A sequence $X=(x_n)$ of real numbers is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists a natural number $H(\varepsilon)$ such that for all natural numbers $n, m \geq H(\varepsilon)$, the terms x_n, x_m satisfy $|x_n - x_m| < \varepsilon$.

Lemma 1.2.1. *If $X = (x_n)$ is a convergent sequence of real numbers, then X is a Cauchy sequence.*

Lemma 1.2.2. *A Cauchy sequence of real numbers is bounded.*

Theorem 1.2.6. Cauchy Convergence Criterion *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

1.3 Introduction to Infinite Series

We will now give a brief introduction to infinite series of real numbers. In elementary texts, an infinite series is sometimes "defined" to be "an expression of the form".

$$x_1 + x_2 + \cdots + x_n + \cdots$$

However, this "definition" lacks clarity, since there is a priori no particular value that we can attach to this array of symbols, which calls for an infinite number of additions to be performed.

Definition 7. If $X := (x_n)$ is a sequence in $\mathbb{R}$, then the **infinite series** (or simply the **series**) **generated by** X is the sequence $S := (s_k)$ defined by

$$\begin{aligned}
&s_1 := x_1 \\
&s_2 := s_1 + x_2 \quad (= x_1 + x_2) \\
&\ldots \\
&s_k := s_{k-1} + x_k \quad (= x_1 + x_2 + \cdots + x_k)
\end{aligned}$$

The numbers x_n are called the **terms** of the series and the numbers

s_k are called the **partial sums** of this series. If $\lim S$ exists, we say that this series is **convergent** and call this limit the **sum** or the **value** of this series. If this limit does not exist, we say that the series S is **divergent**.

It is convenient to use symbols such as

$$\sum (x_n) \text{ or } \sum x_n \text{ or } \sum_{n=1}^{\infty} x_n \tag{1.2}$$

to denote both the infinite series S generated by the sequence $X = (x_n)$ and also to denote the value $\lim S$, in case this limit exists. Thus the symbols in (1.2) may be regarded merely as a way of exhibiting an infinite series whose convergence or divergence is to be investigated. In practice, this double use of these notations does not lead to any confusion, provided it is understood that the convergence (or divergence) of the series must be established.

Just as a sequence may be indexed such that its first element is not x_1, but is x_0, or x_5 or x_{99}, we will denote the series having these numbers as their first element by the symbols

$$\sum_{n=0}^{\infty} x_n \text{ or } \sum_{n=5}^{\infty} x_n \text{ or } \sum_{n=99}^{\infty} x_n$$

It should be noted that when the first term in the series is x_N, then the first partial sum is denoted by s_N.

Note 1.3.1. The words "sequence" and "series" are interchangeable in nonmathematical language. However, in mathematics, these words are not synonyms. Indeed, a series is a sequence $S = (s_k)$ obtained from a given sequence $X = (x_n)$ according to the special procedure given in

Definition 7.

Example 1.3.2.

1. Consider the sequence $X := (r^n)_{n=0}^{\infty}$ where $r \in \mathbb{R}$, which generates the geometric series:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots + r^n + \cdots \tag{1.3}$$

We will show that if $|r| < 1$, then this series converges to $1/(1-r)$. Indeed, if $s_n := 1 + r + r^2 + \cdots + r^n$ for $n \geq 0$, and if we multiply s_n by r and subtract the result from s_n, we obtain (after some simplification):

$$s_n(1-r) = 1 - r^{n+1}.$$

Therefore, we have

$$s_n - \frac{1}{1-r} = -\frac{r^{n+1}}{1-r},$$

from which it follows that

$$\left| s_n - \frac{1}{1-r} \right| \leq \frac{|r|^{n+1}}{|1-r|}$$

Since $|r|^{n+1} \to 0$ when $|r| < 1$, it follows that the geometric series converges to $1/(1-r)$ when $|r| < 1$.

2. Consider the series generated by $((-1)^n)_{n=0}^{\infty}$; that is, the series:

$$\sum_{n=0}^{\infty} (-1)^n = (+1) + (-1) + (+1) + (-1) + \cdots$$

It is easily seen (by Mathematical Induction) that $s_n = 1$ if $n \geq 0$ is even and $s_n = 0$ if n is odd; therefore, the sequence of partial sums is $(1, 0, 1, 0, \ldots)$. Since this sequence is not convergent, the series $\sum_{n=0}^{\infty}(-1)^n$ is divergent.

3. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots$$

By a stroke of insight, we note that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Hence, on adding these terms from $k = 1$ to $k = n$ and noting that the telescoping that takes place, we obtain

$$s_n = \frac{1}{1} - \frac{1}{n+1},$$

whence it follows that $s_n \to 1$. Therefore the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1 .

We now present a very useful and simple necessary condition for the convergence of a series. It is far from being sufficient, however.

The nth Term Test. If the series $\sum x_n$ converges, then $\lim (x_n) = 0$.

Theorem 1.3.3. Cauchy Criterion for Series *The series $\sum x_n$ converges if and only if for every $\varepsilon > 0$ there exists $M(\varepsilon) \in \mathbb{N}$ such that if $m > n \geq M(\varepsilon)$, then*

$$|s_m - s_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \varepsilon.$$

Theorem 1.3.4. *Let (x_n) be a sequence of non negative real numbers. Then the series $\sum x_n$ converges if and only if the sequence $S = (s_k)$ of partial sums is bounded. In this case,*

$$\sum_{n=1}^{\infty} x_n = \lim (s_k) = \sup \{s_k : k \in \mathbb{N}\}$$

Example 1.3.5.

(a) The geometric series $\sum_{n=0}^{\infty} r^n$ diverges if $|r| \geq 1$.

This follows from the fact that the terms r^n do not approach 0 when $|r| \geq 1$.

(b) The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Since the terms $1/n \to 0$, we cannot use the nth Term Test to establish this divergence. This series is famous for the very slow growth of its partial sums. Here we prove the divergence by contradiction. If we assume the series converges to the number S, then we have

$$\begin{aligned} S &= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \cdots + \\ &\quad \left(\frac{1}{2n-1} + \frac{1}{2n}\right) + \cdots \\ &> \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \cdots + \\ &\quad \left(\frac{1}{2n} + \frac{1}{2n}\right) + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \\ &= S. \end{aligned}$$

The contradiction $S > S$ shows the assumption of convergence must be false and the harmonic series must diverge.

(Note that The harmonic series receives its musical name from the fact that the wavelengths of the overtones of a vibrating string are $1/2, 1/3, 1/4, \ldots$, of the string's fundamental wavelength.)

(c) The 2-**series** $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ is convergent.

Since the partial sums are monotone, it suffices to show that some subsequence of (s_k) is bounded. If $k_1 := 2^1 - 1 = 1$, then $s_{k_1} = 1$. If $k_2 := 2^2 - 1 = 3$, then

$$s_{k_2} = \frac{1}{1} + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) < 1 + \frac{2}{2^2} = 1 + \frac{1}{2},$$

and if $k_3 := 2^3 - 1 = 7$, then we have

$$s_{k_3} = s_{k_2} + \left(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right) < s_{k_2} + \frac{4}{4^2} < 1 + \frac{1}{2} + \frac{1}{2^2}.$$

By Mathematical Induction, we find that if $k_j := 2^j - 1$, then

$$0 < s_{k_j} < 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{j-1}.$$

Since the term on the right is a partial sum of a geometric series with $r = \frac{1}{2}$, it is dominated by $1/\left(1 - \frac{1}{2}\right) = 2$. Hence (s_k) is bounded. Then Theorem 1.3.4 implies that the 2 -series converges.

(d) The p-series $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges when $p > 1$.

Since the argument is very similar to the special case considered in part (c), we will leave some of the details to the reader. As before, if $k_1 := 2^1 - 1 = 1$, then $s_{k_1} = 1$. If $k_2 := 2^2 - 1 = 3$, then since $2^p < 3^p$, we have

$$s_{k_2} = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) < 1 + \frac{2}{2^p} = 1 + \frac{1}{2^{p-1}}.$$

Further, if $k_3 := 2^3 - 1$, then it is seen that

$$\begin{aligned} s_{k_3} &= \left(\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) \\ &< s_{k_2} + \frac{4}{4^p} < 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}}. \end{aligned}$$

Finally, we let $r := 1/2^{p-1}$; since $p > 1$, we have $0 < r < 1$. Using Mathematical Induction, we show that if $k_j := 2^j - 1$, then

$$0 < s_{k_j} < 1 + r + r^2 + \cdots + r^{j-1} < \frac{1}{1-r}.$$

Therefore, Theorem 1.3.4 implies that the p-series converges when $p > 1$.

(e) The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges when $0 < p \leq 1$.

We will use the elementary inequality $n^p \leq n$ when $n \in \mathbb{N}$ and $0 < p \leq 1$. It follows that

$$\frac{1}{n} \leq \frac{1}{n^p} \quad \text{for} \quad n \in \mathbb{N}.$$

Since the partial sums of the harmonic series are not bounded, this inequality shows that the partial sums of the p-series are not bounded when $0 < p \leq 1$. Hence the p-series diverges for these values of p.

(f) The alternating harmonic series, given by

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots \tag{1.4}$$

is convergent.

The reader should compare this series with the harmonic series in (b), which is divergent. Thus, the subtraction of some of the terms in (1.4) is essential if this series is to converge. Since we have

$$s_{2n} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right),$$

it is clear that the "even" subsequence (s_{2n}) is increasing. Similarly, the "odd" subsequence (s_{2n+1}) is decreasing since

$$s_{2n+1} = \frac{1}{1} - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \cdots - \left(\frac{1}{2n} - \frac{1}{2n+1}\right).$$

Since $0 < s_{2n} < s_{2n} + 1/(2n+1) = s_{2n+1} \leq 1$, both of these subsequences are bounded below by 0 and above by 1 . Therefore they are both convergent and to the same value. Thus the sequence (s_n) of partial sums converges, proving that the alternating harmonic series (1.4) converges.

Comparison Tests

Our first test shows that if the terms of a non negative series are dominated by the corresponding terms of a convergent series, then the first series is convergent.

Comparison Test. Let $X := (x_n)$ and $Y := (y_n)$ be real sequences and suppose that for some $K \in \mathbb{N}$ we have

$$0 \leq x_n \leq y_n \text{ for } \quad n \geq K \tag{1.5}$$

(a) Then the convergence of $\sum y_n$ implies the convergence of $\sum x_n$.

(b) The divergence of $\sum x_n$ implies the divergence of $\sum y_n$.

Since it is sometimes difficult to establish the inequalities (1.5), the next result is frequently very useful.

Limit Comparison Test. Suppose that $X := (x_n)$ and $Y := (y_n)$ are strictly positive sequences and suppose that the following limit exists in $\mathbb{R}$:

$$r := \lim \left(\frac{x_n}{y_n} \right).$$

(a) If $r \neq 0$ then $\sum x_n$ is convergent if and only if $\sum y_n$ is convergent.

(b) If $r = 0$ and if $\sum y_n$ is convergent, then $\sum x_n$ is convergent.

Remark 1.3.1. The Comparison Test and Limit Comparison Test are depend on having a stock of series that one knows to be convergent (or divergent). We find that the p-series is often useful for this purpose.

Example 1.3.6.

(a) The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges.

It is clear that the inequality

$$0 < \frac{1}{n^2+n} < \frac{1}{n^2} \quad \text{for} \quad n \in \mathbb{N}$$

is valid. Since the series $\sum 1/n^2$ is convergent, we can apply the Comparison Test to obtain the convergence of the given series.

(b) The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-n+1}$ is convergent.

If the inequality

$$\frac{1}{n^2-n+1} \leq \frac{1}{n^2} \tag{1.6}$$

were true, we could argue as in (a). However, (1.6) is false for all $n \in \mathbb{N}$. We have

$$\begin{aligned} &(n-1)^2 + 1 \geq 0 \\ \Rightarrow\ & n^2 - 2n + 2 \geq 0 \\ \Rightarrow\ & 2n^2 - 2n + 2 \geq n^2 \\ \Rightarrow\ & 2(n^2 - n + 1) \geq n^2 \\ \Rightarrow\ & 0 < \frac{1}{n^2-n+1} \leq \frac{2}{n^2} \end{aligned}$$

is valid for all $n \in \mathbb{N}$, and this inequality will work just as well. However, it might take some experimentation to think of such an inequality and then establish it.

Instead, if we take $x_n := 1/\left(n^2 - n + 1\right)$ and $y_n := 1/n^2$, then we have

$$\frac{x_n}{y_n} = \frac{n^2}{n^2 - n + 1} = \frac{1}{1 - (1/n) + (1/n^2)} \to 1$$

Therefore, the convergence of the given series follows from the Limit Comparison Test.

(c) The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ is divergent.

This series closely resembles the series $\sum 1/\sqrt{n}$, which is a p-series with $p = \frac{1}{2}$; it is divergent. If we let $x_n := 1/\sqrt{n+1}$ and $y_n := 1/\sqrt{n}$, then we have

$$\frac{x_n}{y_n} = \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{1}{\sqrt{1 + 1/n}} \to 1.$$

Therefore by Limit Comparison Test both $\sum x_n$ and $\sum y_n$ behaves alike. Since $\sum 1/\sqrt{n}$ is a divergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ is divergent.

(d) The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent.

It would be possible to establish this convergence by showing (by Induction) that $n^2 < n!$ for $n \geq 4$, whence it follows that

$$0 < \frac{1}{n!} < \frac{1}{n^2} \quad \text{for} \quad n \geq 4.$$

Alternatively, if we let $x := 1/n!$ and $y_n := 1/n^2$, then (when $n \geq 4$) we have

$$0 \leq \frac{x_n}{y_n} = \frac{n^2}{n!} = \frac{n}{1 \cdot 2 \cdots (n-1)} < \frac{1}{n-2} \to 0$$

Therefore the Limit Comparison Test implies that $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent. (Note that this test was a bit troublesome to apply since we do not presently know the convergence of any series for which the limit of x_n/y_n is really easy to determine.)

1.4 Limits of Functions

Definition 8. Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a **cluster point** of A if for every $\delta > 0$ there exists at least one point $x \in A, x \neq c$ such that $|x - c| < \delta$.

Example 1.4.1.

1. For the open interval $A_1 := (0, 1)$, every point of the closed interval $[0, 1]$ is a cluster point of A_1. Note that the points 0,1 are cluster points of A_1, but do not belong to A_1. All the points of A_1 are cluster points of A_1

2. A finite set has no cluster points.

3. The infinite set $\mathbb{N}$ has no cluster points.

4. The set $A_4 := \{1/n : n \in \mathbb{N}\}$ has only the point 0 as a cluster point. None of the points in A_4 is a cluster point of A_4.

5. If $I := [0, 1]$, then the set $A_s := I \cap Q$ consists of all the rational numbers in I. Every point in I is a cluster point of A_5.

Definition 9. Let $A \subseteq \mathbb{R}$, and let c be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a limit of f at c if, given any $\varepsilon > 0$, there exists a $d > 0$ such that if $x \in A$ and $0 < |x - c| < d$, then $|f(x) - L| < \varepsilon$.

Definition 10. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a clusterpoint of A. We say that f **is bounded on a neighborhood of** c if there exists a δ-neighborhood $V_\delta(c)$ of c and a constant $M > 0$ such that we have $|f(x)| \leq M$ for all $x \in A \cap V_\delta(c)$.

1.5 Differentiation

Theorem 1.5.1. *If $f : I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.*

Theorem 1.5.2. *Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions that are differentiable at c. Then:*

1. *If $\alpha \in \mathbb{R}$, then the functions that are differentiable at c, and*

$$(\alpha f)'(c) = \alpha f'(c).$$

2. *The function $f + g$ is differentiable at c, and*

$$(f + g)'(c) = f'(c) + g'(c).$$

3. *(Product Rule) The function fg is differentiable at c, and*

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

4. *(Quotient Rule) If $g(c) \neq 0$, then the function f/g is differentiable at c, and*

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

Theorem 1.5.3. Carateodory′s Theorem *Let f be defined on an interval I containing the point c. Then fis differentiable at c if and only if there exists a function φ on I that is continuous at c and satisfies*

$$f(x) - f(c) = \varphi(x)(x - c) \quad \text{for} \quad x \in I$$

In this case, we have $\varphi(c) = f'(c)$.

Theorem 1.5.4. Chain Rule *Let I, J be intervals in $\mathbb{R}$, let $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be functions such that $f(J) \subseteq I$, and let $c \in J$. If f is differentiable at c and if g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and*

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

Theorem 1.5.5. Mean Value Theorem *Suppose that f is continuous on a closed interval $I := [a, b]$, and that f has a derivative in the open interval (a, b). Then there exists at least one point c in (a, b) such that*

$$f(b) - f(a) = f^{'}(c)(b - a).$$

Chapter 2

CONTINUOUS FUNCTIONS, CONTINUOUS FUNCTIONS ON INTERVALS AND UNIFORM CONTINUITY[1]

Here we study continuous functions, continuous functions on interval and uniform continuity.

[1]Syllabus topics begin from this chapter.

2.1 Continuous Functions

In this section, we define continuous function at a point (or on a set), discontinuous function at a point and sequential criterion for continuity and discontinuity

Definition 11. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$. We say that f is **continuous at** c if, given any number $\varepsilon > 0$, there exists $\delta > 0$ such that if x is any point of A satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

If f fails to be continuous at c, then we say that f is **discontinuous at** c .

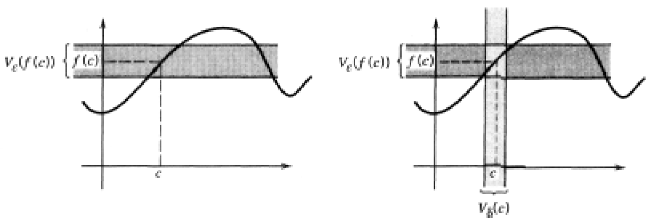

Figure 2.1: Given $V_\varepsilon(f(c))$, a neighborhood $V_\delta(c)$ is to be determined

Remark 2.1.1. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$. If f fails to be continuous at c, then it probably won't be true that for all x, $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$. f is not continuous at c if there exists $\varepsilon > 0$ such that **for all** $\delta > 0$ **there exists** x with $|x - c| < \delta$ and $|f(x) - f(c)| \geq \varepsilon$. So to prove that $f(x)$ is continuous at c you just need to find one x and you need an argument that works for any positive δ.

Example 2.1.1.

1. Let $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x$. Let $\varepsilon > 0$ be given.
   $|f(x) - f(c)| = |x - c| < \varepsilon, \quad$ whenever $|x - c| < \varepsilon$.

Hence taking $\delta = \varepsilon$, we have for any $\varepsilon > 0$ there exists a $\delta > 0$ such that
$|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$.
Therefore for any $c \in \mathbb{R}$ f is continuous at c.

2. Let $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Here f is discontinuous at $c = 0$.
For if, let $\varepsilon = 1$. If $\delta \leq 1$ then $|x - c| = |x - 0| = |x| < \delta$ implies that $|x| < 1$.
So $\frac{1}{|x|} > 1$. That is $|\frac{1}{x} - 0| > \varepsilon$. Hence $\varepsilon = 1$ doesnot satisfy the condition for continuity when $\delta \leq 1$.
If $\delta > 1$ then $|x - c| = |x - 0| = |x| < \delta$. We get $x = 0.5$ satisfies the condition $|x - c| < \delta$ but $|f(0.5) - f(c)| = |2 - 0| = 2 > \varepsilon$. Hence $\varepsilon = 1$ doesnot satisfy the condition for continuity when $\delta > 1$.
Therefore for $\varepsilon = 1$ and for all $\delta > 0$ there exists some $x \in \mathbb{R}$ which doesnot satisfy the definition of continuity. Hence $f(x)$ is discontinuous at 0.

Theorem 2.1.2. *A function $f : A \to \mathbb{R}$ is continuous at a point $c \in A$ if and only if given any ε-neighborhood $V_\varepsilon(f(c))$ of $f(c)$ there exists a δ-neighborhood $V_\delta(c)$ of c such that if x is any point of $A \cap V_\delta(c)$, then $f(x)$ belongs to $V_s(f(c))$, that is,*

$$f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c))$$

Remark 2.1.2.

1. If $c \in A$ is a cluster point of A, then a comparison of Definitions of limit and continuity show that f is continuous at c if and only if
$$f(c) = \lim_{x \to c} f(x)$$
Thus, if c is a cluster point of A, then three conditions must hold for f to be continuous at c :
   (i) f must be defined at c (so that $f(c)$ makes sense),
   (ii) the limit of f at c must exist in $\mathbb{R}$ (so that $\lim_{x \to c} f(x)$ makes sense), and
   (iii) these two values must be equal.
2. If $c \in A$ is not a cluster point of A, then there exists a neighborhood $V_\delta(c)$ of c such that $A \cap V_\delta(c) = \{c\}$. Thus we conclude that a function f is automatically continuous at a point $c \in A$ that is not a cluster point of A. Such points are often called **isolated points** of A. They are of little practical interest to us, since they have no relation to a limiting process. Since continuity is automatic for such points, we generally test for continuity only at cluster points. Thus we regard condition (1) as being characteristic for continuity at c.

Definition 12. (**Sequential Criterion for Continuity**) A function $f : A \to \mathbb{R}$ is continuous at the point $c \in A$ if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.

Definition 13. (**Discontinuity Criterion**) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$. Then f is discontinuous at c if and only if there exists a sequence (x_n) in A such that (x_n) converges to c, but the sequence $(f(x_n))$ does not converge to $f(c)$.

So far we have discussed continuity at a point. To talk about the continuity of a function on a set, we will simply require that the function be continuous at each point of the set. We state this formally in the next definition.

Definition 14. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If B is a subset of A, we say that f is **continuous on the set** B if f is continuous at every point of B.

Example 2.1.3.

(a) The constant function $f(x) := b$ is continuous on $\mathbb{R}$, where $b \in \mathbb{R}$. If $c \in \mathbb{R}$, then $\lim\limits_{x \to c} f(x) = b$. Since $f(c) = b$, we have $\lim\limits_{x \to c} f(x) = f(c)$, and thus f is continuous at every point $c \in \mathbb{R}$. Therefore f is continuous on $\mathbb{R}$.

(b) $g(x) := x$ is continuous on $\mathbb{R}$.

If $c \in \mathbb{R}$, then we have $\lim\limits_{x \to c} g(x) = c$. Since $g(c) = c$, then g is continuous at every point $c \in \mathbb{R}$. Thus g is continuous on $\mathbb{R}$.

(c) $h(x) := x^2$ is continuous on $\mathbb{R}$.
If $c \in \mathbb{R}$, then we have $\lim\limits_{x \to c} h(x) = c^2$. Since $h(c) = c^2$, then h is continuous at every point $c \in \mathbb{R}$. Thus h is continuous on $\mathbb{R}$.

(d) $\varphi(x) := \frac{1}{x}$ is continuous on $A := \{x \in \mathbb{R} : x > 0\}$.
If $c \in A$, then we have $\lim\limits_{x \to c} \varphi(x) = \frac{1}{c}$. Since $\varphi(c) = \frac{1}{c}$, this shows that φ is continuous at every point $c \in A$. Thus φ is continuous on A.

(e) $\varphi(x) := \frac{1}{x}$ is not continuous at $x = 0$.

Indeed, if $\varphi(x) = \frac{1}{x}$ for $x > 0$, then φ is not defined for $x = 0$, so it cannot be continuous there. Alternatively, $\lim\limits_{x \to 0} \varphi(x)$ does not exist in $\mathbb{R}$, so φ cannot be continuous at $x = 0$.

(f) The signum function sgn is defined by

$$sgn(x) = \begin{cases} +1 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -1 & \text{for } x < 0. \end{cases}$$

is not continuous at 0.

(g) Let $A := \mathbb{R}$ and let f be Dirichlet's "discontinuous function" defined by

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } \quad x \text{ is irrational} \end{cases}$$

We claim that f is not continuous at any point of $\mathbb{R}$. (This function was introduced in 1829 by **P. G. L. Dirichlet**.)

Indeed, if c is a rational number, let (x_n) be a sequence of irrational numbers that converges to c (Corollary 1.1.1 to the Density Theorem 1.1.3 assures us that such a sequence does exist). Since $f(x_n) = 0$ for all $n \in \mathbb{N}$, we have $\lim (f(x_n)) = 0$, while $f(c) = 1$. Therefore f is not continuous at the rational number c.

On the other hand, if b is an irrational number, let (y_n) be a sequence of rational numbers that converge to b. (The Density Theorem 1.1.3 assures us that such a sequence does exist). Since $f(y_n) = 1$ for all $n \in \mathbb{N}$, we have $\lim (f(y_n)) = 1$, while $f(b) = 0$. Therefore f is not continuous at the irrational number b.

Since every real number is either rational or irrational, we deduce that f is not continuous at any point in $\mathbb{R}$.

(h) Let $A := \{x \in \mathbb{R} : x > 0\}$. For any irrational number $x > 0$ we define $h(x) := 0$. For a rational number in A of the form m/n, with natural numbers m, n having no common factors except 1, we define $h(m/n) := 1/n$ (see Figure(**??**)). *ie*,

$$h(x) = \begin{cases} 0, & if\ x = 0\ or\ x\ is\ an\ irrational\ number \\ \frac{1}{n}, & if\ x = \frac{m}{n}\ where\ m, n \in \mathbb{N}\ and\ gcd(m, n) = 1 \end{cases}$$

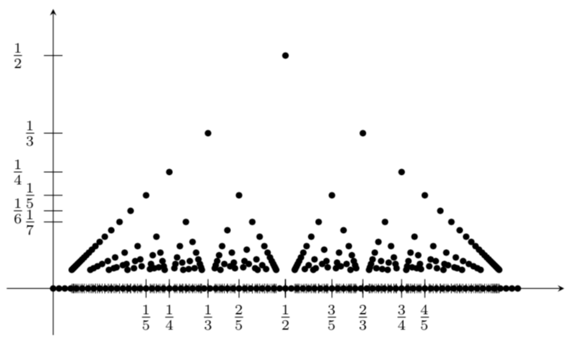

Figure 2.2:

We claim that h is continuous at every irrational number in A, and is discontinuous at every rational number in A. (This function was introduced in 1875 by K. J. Thomae.) Indeed, if $a > 0$ is rational, let (x_n) be a sequence of irrational numbers in A that converges to a. Then $\lim(h(x_n)) = 0$, while $h(a) > 0$. Hence h is discontinuous at a. On the other hand, if b is an irrational number and $\varepsilon > 0$, then (by the Archimedean Property) there is a natural number n_0 such that $1/n_0 < \varepsilon$. There are only a finite number of rationals with denominator less than n_0 in the interval $(b-1, b+1)$. Hence $\delta > 0$ can be chosen so small that the neighborhood $(b-\delta, b+\delta)$ contains no rational numbers with denominator less than n_0. It then follows that for $|x-b| < \delta$, $x \in A$, we have $|h(x) - h(b)| = |h(x)| \leq 1/n_0 < \varepsilon$. Thus h is continuous at the irrational number b. Consequently, we deduce that Thomae's function h is continuous precisely at the irrational points in A.

(i) Let $f(x) = x\sin(1/x)$ for $x \neq 0$. Since f is not defined at $x = 0$, the function f cannot be continuous at this point. However, $\lim\limits_{x\to 0}(x\sin(1/x)) = 0$. If we define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \begin{cases} 0, & \text{for } x = 0 \\ x\sin(1/x), & \text{for } x \neq 0 \end{cases}$$

then F is continuous at $x = 0$.

(j) The function $g(x) := \sin(1/x)$ for $x \neq 0$ does not have a limit at $x = 0$. Thus there is no value that we can assign at $x = 0$ to obtain a continuous extension of g at $x = 0$.

EXERCISES

1. Let $a < b < c$. Suppose that f is continuouson $[a, b]$, that g is continuous on $[b, c]$, and that $f(b) = g(b)$. Define h on $[a, c]$ by $h(x) := f(x)$ for $x \in [a, b]$ and $h(x) := g(x)$ for $x \in (b, c]$. Prove that h is continuous on $[a, c]$.

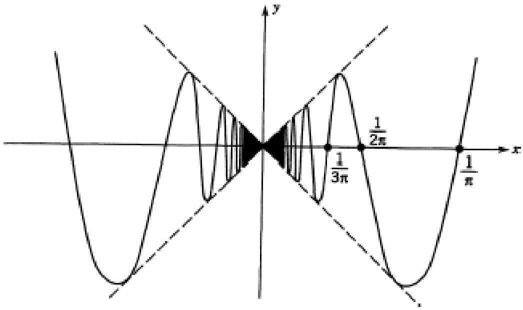

Figure 2.3: Graph of $f(x) = x\sin(1/x) \quad (x \neq 0)$

2. If $x \in \mathbb{R}$, we define $\lceil x \rceil$ to be the greatest integer $n \in \mathbb{Z}$ such that $n \leq x$. (Thus, for example, $\lceil 8.3 \rceil = 8$, $\lceil \pi \rceil = 3$, $\lceil -\pi \rceil = -4$.) The function $x \to \lceil x \rceil$ is called the **greatest integer function**. Determine the points of continuity of the following functions:

   (*a*) $f(x) := \lceil x \rceil$, (*b*) $g(x) := x\lceil x \rceil$,

   (*c*) $h(x) := \lceil \sin x \rceil$, (*d*) $k(x) := \lceil 1/x \rceil$ $(x \neq 0)$.

3. Let f be defined for all $x \in \mathbb{R}$, $x \neq 2$, by $f(x) = (x^2+x-6)/(x-2)$. Can f be defined at $x = 2$ in such a way that f is continuous at this point?

4. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be continuous at a point $c \in A$. Show that for any $\varepsilon > 0$, there exists a neighborhood $V_\delta(c)$ of c such that if $x, y \in A \cap V_\delta(c)$, then $|f(x) - f(y)| < \varepsilon$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at c and let $f(c) > 0$. Show that there exists a neighborhood $V_\delta(c)$ of c such that if $x \in V_\delta(c)$, then $f(x) > 0$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $S := \{x \in \mathbb{R} : f(x) = 0\}$ be the "zero set" of f. If (x_n) is in S and $x = \lim(x_n)$, show that $x \in S$.

7. Let $A \subseteq B \subseteq \mathbb{R}$, let $f : B \to \mathbb{R}$ and let g be the restriction of f to A (that is, $g(x) = f(x)$ for $x \in A$).

   (a) If f is continuous at $c \in A$, show that g is continuous at c.

   (b) Show by example that if g is continuous at c, it need not follow that f is continuous at c.

8. Show that the absolute value function $f(x) := |x|$ is continuous at every point $c \in \mathbb{R}$.

9. Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$. Show that f is continuous at every point $c \in \mathbb{R}$.

10. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number r. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

11. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := 2x$ for x rational, and $g(x) := x + 3$ for x irrational. Find all points at which g is continuous.

2.2 Continuous Functions on Intervals

Functions that are continuous on intervals have a number of very important properties that are not possessed by general continuous functions. In this section, we will establish some deep results that are of considerable importance.

Definition 15. A function $f : A \to \mathbb{R}$ is said to be **bounded on** A if there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$ and f is **unbounded on** A if given any $M > 0$, there exists a point $x_M \in A$ such that $|f(x_M)| > M$.

Example 2.2.1. the function f defined on the interval $A := (0, \infty)$ by $f(x) := 1/x$ is not bounded on A because for any $M > 0$ we can take the point $x_M := 1/(M+1)$ in A to get $f(x_M) = 1/x_M = M + 1 > M$.

Note 2.2.2. Example (2.2.1) shows that continuous functions need not be bounded.

Theorem 2.2.3. Boundedness Theorem *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on I.*

Proof. Suppose that I is not bounded on I. Then, for any $n \in N$ there is a number $x_n \in I$ such that $|f(x_n)| > n$. Since I is bounded, the sequence $X := (x_n)$ is bounded. Therefore, the Bolzano-WeierstrassTheorem 1.2.5 implies that there is a subsequence $X' = (x_{n_r})$ of X that converges to a number x. Since I is closed and the elements of X' belong to I, it follows from Theorem 1.2.3 that $x \in I$. Then I is continuous at x, so that $f(x_n)$ converges to $f(x)$. We then conclude from Theorem 1.2.1 that the convergent sequence $f(x_n)$ must be bounded. But this is a contradiction since

$$|f(x_{n_r})| > n_r > r \qquad for\ r \in \mathbb{N}$$

Therefore the supposition that the continuous function f is not bounded on the closed bounded interval I leads to a contradiction. $\square$

Example 2.2.4. Each hypothesis of the Boundedness Theorem is needed, the following examples shows that the conclusion fails if anyone of the hypotheses is relaxed.

(i) The interval must be bounded. The function $f(x) := x$ for x in the unbounded, closed interval $A := [0, \infty)$ is continuous but not bounded on A.

(ii) The interval must be closed. The function $g(x) := 1/x$ for x in the half-open interval $B := (0, 1]$ is continuous but not bounded on B.

(iii) The function must be continuous. The function h defined on the closed interval $C := [0, 1]$ by $h(x) := 1/x$ for $x \in (0, 1]$ and $h(0) := 1$ is discontinuous and unbounded on C.

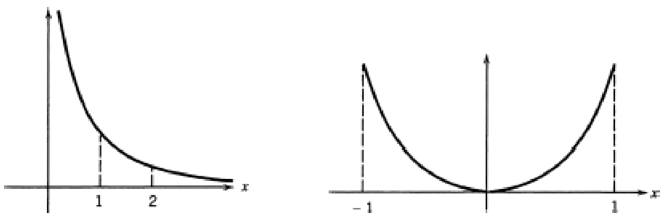

Figure 2.4: Functions $f(x) = 1/x \quad (x > 0)$ and $g(x) = x^2 \quad (|x| \leq 1)$

The Maximum-Minimum Theorem

Definition 16. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that f **has an absolute maximum** on A if there is a point $x^* \in A$ such that

$$f(x^*) \geq f(x) \quad \textit{for all } x \in A$$

We say that f **has an absolute minimum** on A if there is a point $x_* \in A$ such that

$$f(x_*) \leq f(x) \quad \text{for all } x \in A$$

We say that x^* is an **absolute maximum point** for f on A, and that x_* is an **absolute minimum point** for f on A, if they exist.

Note 2.2.5. A continuous function on a set A does not necessarily have an absolute maximum or an absolute minimum on the set.

Example 2.2.6.

1. Let $A := (0, \infty)$ and let $f : A \to \mathbb{R}$ is defined by

$$f(x) = 1/x$$

   Since f is not bounded above on A, there can be no absolute maximum for f on A. There is no point at which f attains the value $0 = \inf \{f(x) : x \in A\}$. So there can be no absolute minimum for f on A. Therefore $f(x) := 1/x$ has neither an absolute maximum nor an absolute minimum on the set $A := (0, \infty)$.

2. Let $B := (0, 1)$ and let $f : B \to \mathbb{R}$ is defined by

$$f(x) = 1/x$$

   Since f is not bounded above on B, there can be no absolute maximum for f on B. There is no point at which f attains the value $1 = \inf \{f(x) : x \in B\}$. So there can be no absolute minimum for f on B. Therefore $f(x) := 1/x$ has neither an absolute maximum nor an absolute minimum on the set B. While it has both an absolute maximum and an absolute minimum when it is restricted to the set $[1, 2]$. In addition, $f(x) = 1/x$ has an absolute maximum but no absolute minimum when restricted to the set $[1, \infty)$, but no absolute maximum and no absolute minimum when restricted to the set $(1, \infty)$.

3. if a function has an absolute maximum point, then this point is not necessarily uniquely determined. The function $g(x) := x^2$ defined for $x \in A := [-1, +1]$ has the two points $x = \pm 1$ giving the absolute maximum on A, and the single point $x = 0$ yielding its absolute minimum on A.

4. The constant function $h(x) := 1$ for $x \in \mathbb{R}$ is such that every point of $\mathbb{R}$ is both an absolute maximum and an absolute minimum point for h.

Theorem 2.2.7. Maximum – Minimum Theorem *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I.*

Proof. Consider the nonempty set $f(I) := \{f(x) : x \in I\}$ of values of f on I. In Boundedness Theorem (2.2.3) it was established that $f(I)$ is a bounded subset of $\mathbb{R}$. Let $s^* := \sup f(I)$ and $s_* := \inf f(I)$.
Claim1 : There exist point x^* in I such that $s^* = f(x^*)$.
Since $s^* = \sup f(I)$, if $n \in \mathbb{N}$, then the number $s^* - 1/n$ is not an upper bound of the set $f(I)$. Consequently there exists a number $x_n \in I$ such that

$$s^* - 1/n < f(x_n) \leq s^* \qquad for\ all\ n \in \mathbb{N} \tag{2.1}$$

Since I is bounded, the sequence $X := (x_n)$ is bounded. Therefore, by the Bolzano- Weierstrass Theorem 1.2.5 there is a subsequence $X' = (x_{n_r})$ of X that converges to some number x^*, that is $\lim(x_{n_r}) = x^*$.

Since the elements of X' belong to $I = [a, b]$, we write $a \leq x_{n_r} \leq b$. From Theorem 1.2.3 we get $a \leq \lim(x_{n_r}) \leq b$. That is, $x^* \in I$. Given that f is continuous on I. Therefore f must be continuous at x^*. So $\lim f(x_{n_r}) = f(x^*)$. Since it follows from equation (2.1) that

$$s^* - 1/(n_r) < f(x_{n_r}) \leq s^* \qquad \textit{for all } r \in \mathbb{N}$$

We conclude from the Squeeze Theorem 1.2.4 that $\lim f(x_{n_r}) = s^*$. Therefore we have

$$f(x^*) = \lim(f(x_{n_r})) = s^* = \sup f(I)$$

We conclude that x^* is an absolute maximum point of f on I. Hence we proved claim1.

Claim2 : There exist point x_* in I such that $s_* = f(x_*)$.
Since $s_* = \inf f(I)$, if $n \in \mathbb{N}$, then the number $s_* + 1/n$ is not a lower bound of the set $f(I)$. Consequently there exists a number $y_n \in I$ such that

$$s_* \leq f(y_n) < s_* + 1/n \qquad \textit{for all } n \in \mathbb{N} \tag{2.2}$$

Since I is bounded, the sequence $Y := (y_n)$ is bounded. Therefore, by the Bolzano- Weierstrass Theorem 1.2.5 there is a subsequence $Y' = (y_{n_r})$ of Y that converges to some number x_*, that is $\lim(y_{n_r}) = x_*$. Since the elements of Y' belong to $I = [a, b]$, we write $a \leq y_{n_r} \leq b$. From Theorem 1.2.3 we get $a \leq \lim(y_{n_r}) \leq b$. That is, $x_* \in I$. Given that f is continuous on I. Therefore f must be continuous at x_*. So $\lim f(y_{n_r}) = f(x_*)$. Since it follows from equation (2.2) that

$$s_* \leq f(y_{n_r}) < s_* + 1/(n_r) \qquad \textit{for all } r \in \mathbb{N}$$

We conclude from the Squeeze Theorem 1.2.4 that $\lim f(y_{n_r}) = s_*$. Therefore we have

$$f(x_*) = \lim(f(y_{n_r})) = s_* = \inf f(I)$$

We conclude that x_* is an absolute minimum point of f on I. Hence we proved claim2.
Hence f has an absolute maximum and an absolute minimum on I. $\square$

The next result is the theoretical basis for locating roots of a continuous function by means of sign changes of the function. The proof also provides an algorithm, known as the **Bisection Method**, for the calculation of roots to a specified degree of accuracy and can be readily programmed for a computer. It is a standard tool for finding solutions of equations of the form $f(x) = 0$, where f is a continuous function.

Theorem 2.2.8. Location of Roots Theorem *Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 < f(b)$, or if $f(a) > 0 > f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = 0$.*

Proof. We assume that $f(a) < 0 < f(b)$. We will generate a sequence of intervals by successive bisections. Let $I_1 := [a_1, b_1]$ where $a_1 := a,\ b_1 := b$, and let P_1 be the midpoint $P_1 := \frac{1}{2}(a_1+b_1)$. If $f(P_1) = 0$, we take $c := P_1$ and we are done. If $f(P_1) \neq 0$, then either $f(P_1) > 0$ or $f(P_1) < 0$. If $f(P_1) > 0$, then we set $a_2 := a_1,\ b_2 := P_1$, while if $f(P_1) < 0$, then we set $a_2 := P_1, b_2 := b_1$. In either case, we let $I_2 := [a_2, b_2]$; then we have $I_2 \subset I_1$ and $f(a_2) < 0,\ f(b_2) > 0$. We continue the bisection process. Suppose that the intervals $I_1, I_2, ..., I_k$ have been obtained by successive bisection in the same manner. Then we have $f(a_k) < 0$ and $f(b_k) > 0$, and we set $Pk := \frac{1}{2}(a_k + b_k)$. If $f(P_k) = 0$, we take $c := P_k$ and we are

done. If $f(P_k) > 0$, we set $a_{k+1} := a_k$, $b_{k+1} := P_k$ while if $f(P_k) < 0$, we set $a_{k+1} := P_k$, $b_{k+1} := b_k$. In either case, we let $I_{k+1} := [a_{k+1}, b_{k+1}]$; then $I_{k+1} \subset I_k$ and $f(a_{k+1}) < 0,\ f(b_{k+1}) > 0$.
If the process terminates by locating a point P_n such that $f(p_n) = 0$, then we are done. If the process does not terminate, then we obtain a nested sequence of closed bounded intervals $I_n := [a_n, b_n]$ such that for every $n \in \mathbb{N}$ we have

$$f(a_n) < 0 \textit{ and } f(b_n) > 0$$

Furthermore, since the intervals are obtained by repeated bisection, the length of I_n is equal to $b_n - a_n = (b-a)/2^{n-1}$. It follows from the Nested Intervals Property 2.5.2 that there exists a point c that belongs to I_n for all $n \in \mathbb{N}$. Since $a_n \leq c \leq b_n$ for all $n \in \mathbb{N}$ and $\lim(b_n - a_n) = 0$, it follows that $\lim(a_n) = c = \lim(b_n)$. Since f is continuous at c, we have

$$\lim(f(a_n)) = f(c) = \lim(f(b_n))$$

The fact that $f(a_n) < 0$ for all $n \in \mathbb{N}$ implies that $f(c) = \lim(f(a_n)) \leq 0$. Also, the fact that $f(b_n) > 0$ for all $n \in \mathbb{N}$ implies that $f(c) = \lim(f(b_n)) \geq 0$. Thus, we conclude that $f(c) = 0$. Consequently, c is a root of f. $\square$

Theorem 2.2.9. Bolzano′s Intermediate Value Theorem *Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then there exists a point $c \in I$ between a and b such that $f(c) = k$.*

Proof. Suppose that $a < b$ and let $g(x) := f(x) - k$; then $g(a) < 0 < g(b)$. Since f is continuous, g is continuous. By the - Location of Roots

Theorem 2.2.8 there exists a point c with $a < c < b$ such that $g(c) = 0$. That is $f(c) - k = 0$. Therefore $f(c) = k$.
If $b < a$, let $h(x) := k - f(x)$ so that $h(b) < 0 < h(a)$. Therefore there exists a point c with $b < c < a$ such that $h(c) = 0$. That is $k - f(c) = 0$ whence $f(c) = k$. □

Corollary 2.2.1. *Let $I = [a, b]$ be a closed, bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. If $k \in \mathbb{R}$ is any number satisfying*

$$\inf f(I) < k < \sup f(I),$$

then there exists a number $c \in I$ such that $f(c) = k$.

Proof. It follows from the Maximum-MinimumTheorem 2.2.7 that there are points c^* and c_* in I such that $\inf f(I) = f(c_*)$ and $f(c^*) = \sup f(I)$. So $f(c_*) < k < f(c^*)$. Then by Bolzano's Intermediate Value Theorem 2.2.9 there exists a number $c \in I$ such that $f(c) = k$. Hence we proved. □

Theorem 2.2.10. *Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I) := \{f(x) : x \in I\}$ is a closed bounded interval.*

Proof. If we let $m := \inf f(I)$ and $M := \sup f(I)$, then we know from the Maximum- Minimum Theorem 2.2.7 that m and M belong to $f(I)$. Moreover, we have $f(I) \subset [m, M]$. If k is any element of $[m, M]$, then it follows from the preceding corollary 2.2.1 that there exists a point $c \in I$ such that $k = f(c)$. Hence, $k \in f(I)$ and we conclude that $[m, M] \subset f(I)$. Therefore, $f(I)$ is the interval $[m, M]$. Hence $f(I)$ is closed and bounded interval. □

Note 2.2.11.

1. Image of a closed bounded interval under a continuous function is also a closed bounded interval.

2. The endpoints of the image interval are the absolute minimum and absolute maximum values of the real valued function defined on an interval.

3. If $I := [a, b]$ is an interval and $f : I \to \mathbb{R}$ is continuous on I, we have proved that $f(I)$ is the interval $[m, M]$. We have not proved (and it is not always true) that $f(I)$ is the interval $[f(a), f(b)]$. In Figure 2.5 $f(I) = [0, 25] = [m, M]$ but $[f(a), f(b)] = [4, 25]$. So $f(I) \neq [f(a), f(b)]$.

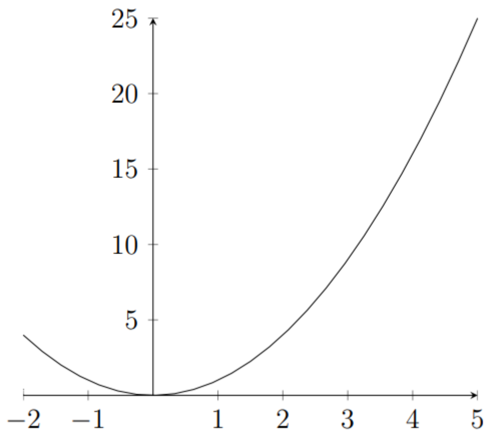

Figure 2.5:

Theorem 2.2.12. Preservation of Intervals Theorem *Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I)$ is an interval.*

Proof. Let $\alpha, \beta \in f(I)$ with $\alpha < \beta$; then there exist points $a, b \in I$ such that $\alpha = f(a)$ and $\beta = f(b)$. Further, it follows from Bolzano's Intermediate Value Theorem 2.2.9 that if $k \in (\alpha, \beta)$ then there exists a number $c \in I$ with $k = f(c) \in f(I)$. Therefore $[\alpha, \beta] \subset f(I)$, showing that $f(I)$ possesses property (1.1) of Theorem 1.1.4. Therefore $f(I)$ is an interval. □

Note 2.2.13.

1. Theorem 2.2.10 states that the continuous image of a closed bounded interval is a set of the same type.

2. Theorem 2.2.12 extends this result to general intervals. Continuous image of an interval is an interval.

3. It is not true that the image interval necessarily has the same form as the domain interval. For example, the continuous image of an open interval need not be an open interval, and the continuous image of an unbounded closed interval need not be a closed interval. Indeed, if $f(x) := 1/(x^2 + 1)$ for $x \in \mathbb{R}$, then f is continuous on $\mathbb{R}$. It is easy to see that if $I_1 := (-1, 1)$, then $f(I_1) = (1/2, 1]$. Here I_1 is an open interval but $f(I_1)$ is not an open interval. Also, if $I_2 := [0, \infty)$, then $f(I_2) = (0, 1]$. Here I_2 is unbounded closed interval but $f(I_2)$ is not a closed interval. (See Figure 2.6.)

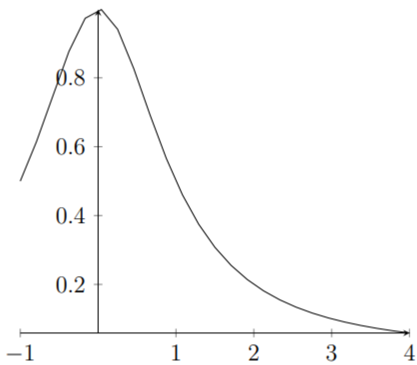

Figure 2.6:

EXERCISES

1. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for each x in I. Prove that there exists a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in I$.

2. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be continuous functions on I. Show that the set $E := \{x \in I : f(x) = g(x)\}$ has the property that if $(x_n) \subset E$ and $x_n \to x_0$ then $X_0 \in E$. [Hint: Use sequential criteria for continuous functions.]

3. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function on I such that for each x in I there exists y in I such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Prove there exists a point c in I such that $f(c) = 0$.

4. Show that every polynomial of odd degree with real coefficients has at least one real root.

5. Show that the polynomial $p(x) := x^4 + 7x^3 - 9$ has at least two real roots. Use a calculator to locate these roots to within two decimal places.

6. Let f be continuous on the interval $[0, 1]$ to $\mathbb{R}$ and such that $f(0) = f(1)$. Prove that there exists a point c in $[0, 4]$ such that $f(c) = f(c + 4)$.
[Hint:Consider $g(x) = f(x) - f(x + 4)$.]
Conclude that there are, at any time, antipodal points on the earth's equator that have the same temperature.

7. Show that the equation $x = \cos x$ has a solution in the interval $[0, \pi/2]$.

8. Show that the function $f(x) := 2 \ln x + \sqrt{x} - 2$ has root in the interval $[1, 2]$.

9. Let $I := [a, b]$, let $f : I \to \mathbb{R}$ be continuous on I, and assume that $f(a) < 0$, $f(b) > 0$. Let $W := \{x \in I : f(x) < 0\}$, and let $w := \sup W$. Prove that $f(w) = 0$. (This provides an alternative proof of Theorem 2.2.8.)

10. Let $I := [0, \pi/2]$ and let $f : I \to \mathbb{R}$ be defined by $f(x) := \sup\{x^2, \cos x\}$ for $x \in I$. Show there exists an absolute minimum point $x_0 \in I$ for f on I. Show that x_0 is a solution to the equation $\cos x = x^2$.

11. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim\limits_{x\to-\infty} f(x) = 0$ and
$\lim\limits_{x\to\infty} f(x) = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or minimum on $\mathbb{R}$. Give an example to show that both a maximum and a minimum need not be attained.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $\beta \in \mathbb{R}$. Show that if $x_0 \in \mathbb{R}$ is such that $f(x_0) < \beta$, then there exists a δ-neighborhood U of x_0 such that $f(x) < \beta$ for all $x \in U$.

13. Examine which open [respectively,closed] intervals are mapped by $f(x) := x^2$ for $x \in \mathbb{R}$ onto open [respectively, closed] intervals.

14. Examine the mapping of open [respectively, closed] intervals under the functions $g(x) := 1/(x^2 + 1)$ and $h(x) := x^3$ for $x \in \mathbb{R}$.

15. If $f : [0, 1] \to \mathbb{R}$ is continuous and has only rational [respectively, irrational] values, must f be constant? Prove your assertion. .

16. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a (not necessarily continuous) function with the property that for every $x \in I$, the function f is bounded on a neighborhood $V_{\delta_x}(x)$ of x (in the sense of ; Definition 4.2.1). Prove that f is bounded on I.

17. Let $J := (a, b)$ and let $g : J \to \mathbb{R}$ be a continuous function with the property that for every $x \in J$, the function g is bounded on a neighborhood $V_{\delta_x}(x)$ of x. Show by example that g is not necessarily bounded on J.

2.3 Uniform Continuity

Definition 17. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that f is **uniformly continuous** on A if for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $x, u \in A$ are any numbers satisfying $|x - u| < \delta(\varepsilon)$, then $|f(x) - f(u)| < \varepsilon$.

In Definition 11 we can change the notation of δ by $\delta(\varepsilon, u)$ or we can say that for $A \subseteq \mathbb{R}$, a function $f : A \to \mathbb{R}$ is continuous at every point $u \in A$ if and only if for any given $\varepsilon > 0$ and $u \in A$, there is a $\delta(\varepsilon, u) > 0$ such that for all x such that $x \in A$ and $|x - u| < \delta(\varepsilon, u)$, then $|f(x) - f(u)| < \varepsilon$. Here $\delta(\varepsilon, u)$ emphasize that it depends on ε and u. The fact that δ depends on u is a reflection of the fact that the function f may change its values rapidly near certain points and slowly near other points.

Now it often happens that the function f is such that the number δ can be chosen to be independent of the point $u \in A$ and to depend only on ε. If δ independent on u it is possible to guarantee that $f(x)$ and $f(u)$ be as close to each other as we please by requiring only that x and y be sufficiently close to each other. Such functions are called uniformly continuous functions on A. For example, if $f(x) := 2x$ for all $x \in \mathbb{R}$, then

$$|f(x) - f(u)| = |2x - 2u| = 2|x - u|,$$

and so we can choose $\delta(\varepsilon, u) := \frac{\varepsilon}{2}$ for all $\varepsilon > 0, u \in \mathbb{R}$. So here $\delta(\varepsilon, u)$ is independent of u and $\delta(\varepsilon, u)$ can be replaced by $\delta(\varepsilon)$.

Note 2.3.1.

(a) If f is uniformly continuous on A, then it is continuous at every point of A.

(b) If f is continuous on every point of a set A doesnot imply that f is uniformly continuous on A.

(c) $g(x) = 1/x$ on the set $A := \{x \in \mathbb{R} : x > 0\}$ is continuous but not uniformly continuous.

Nonuniform Continuity Criteria Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Then the following statements are equivalent:

(i) f is not uniformly continuous on A.

(ii) There exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are points x_δ, u_δ in A such that $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$

(iii) There exists an $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in A such that $\lim(x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

Question: 1. Show that $g(x) := 1/x$ is not uniformly continuous on $A := \{x \in \mathbb{R} : x > 0\}$.

Answer: 1. Let $x_n := \frac{1}{n}$ and $u_n := \frac{1}{n+1}$ [both sequences are in A], then we have
$\lim(x_n - u_n) = \lim(\frac{1}{n} - \frac{1}{n+1}) = \lim(\frac{1}{n(n+1)}) = 0$,
but $|g(x_n) - g(u_n)| = |\frac{1}{x_n} - \frac{1}{u_n}| = |n - (n+1)| = 1$ for all $n \in \mathbb{N}$.
Now choose $\varepsilon_0 = 1$. So there exists an $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in A such that $\lim(x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| = \varepsilon_0$ for all $n \in \mathbb{N}$. Therefore by above equivalent conditions g is not uniformly continuous on A.

Theorem 2.3.2. Uniform Continuity Theorem *Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.*

Proof. If f is not uniformly continuous on I then, by the preceding result, there exists $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in I such that $|x_n - u_n| < 1/n$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$. Since I is bounded, the sequence (x_n) is bounded; by the Bolzano- Weierstrass Theorem 1.2.5 there is a subsequence (x_{n_k}) of (x_n) that converges to an element z. Since I is closed, the limit z belongs to I, by Theorem 1.2.3. It is clear that the corresponding subsequence (u_{n_k}) also converges to z, since $|u_{n_k} - z| \leq |u_{n_k} - x_{n_k}| + |x_{n_k} - z|$. Now if f is continuous at the point z, then both of the sequences $(f(x_{n_k}))$ and $(f(u_{n_k}))$ must converge to $f(z)$. But this is not possible since

$$|f(x_n) - f(u_n)| \geq \varepsilon_0$$

for all $n \in \mathbb{N}$. Thus the hypothesis that f is not uniformly continuous on the closed bounded interval I implies that f is not continuous at some point $z \in I$. Consequently, if f is continuous at every point of I, then f is uniformly continuous on I. $\square$

Lipschitz Functions

If a uniformly continuous function is given on a set that is not a closed bounded interval, then it is sometimes difficult to establish its uniform continuity. However, there is a condition that frequently occurs that is sufficient to guarantee uniform continuity.

Definition 18. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that

$$|f(x) - f(u)| < K|x - u|$$

for all $x, u \in A, x \neq u$, then f is said to be a **Lipschitz function** (or to satisfy a **Lipschitz condition**) on A.

Note 2.3.3. Let $f : I \to \mathbb{R}$ on an interval I is a lipschitz function then,

$|f(x) - f(u)| \leq K|x - u|$ for all $x, u \in I$.

$\Rightarrow \left|\frac{f(x)-f(u)}{x-u}\right| \leq K$, $x, u \in I, x \neq u$.

Then the quantity inside the absolute values is the slope of a line segment joining the points $(x, f(x))$ and $(u, f(u))$. Thus a function f satisfies a Lipschitz condition if and only if the slopes of all line segments joining two points on the graph of $y = f(x)$ over I are bounded by some number K.

Theorem 2.3.4. *If $f : A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.*

Proof. Assume that f is a lipschitz function on A then for $x, u \in A$ we have
$|f(x) - f(u)| < K|x - u|$ for some $K > 0$.
So $|x - u| < \delta \Rightarrow |f(x) - f(u)| < K\delta$
Now choose $K\delta = \varepsilon$. ie, $\delta = \frac{\varepsilon}{K}$. Here δ depend on ε and not on $x \in A$.
Hence we can say that for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $x, u \in A$ are any numbers satisfying $|x-u| < \delta(\varepsilon)$, then $f(x)-f(u)| < \varepsilon$.
Then f is uniformly continuous on A. □

Example 2.3.5.

1. If $f(x) = x^2$ on $A = [0, b]$, where $b > 0$, then

$$\begin{aligned} |f(x) - f(u)| &= |x^2 - u^2| = |(x - u)(x + u)| \\ &= |x - u||x + u| \le (|x| + |u|)|x - u| \\ &\le 2b|x - u| \end{aligned}$$

   That is f satisfies Lipschitz condition on A. Hence f is uniformly continuous on A.

2. Every uniformly continuous function need not be Lipschitz function.

Let $g(x) := \sqrt{x}$ for x in the closed bounded interval $I := [0, 2]$. Since g is continuous on I, it follows from the Uniform Continuity Theorem 2.3.2 that g **is uniformly continuous on** I.

Assume that there exists a $K > 0$ such that $|g(x)| \leq K|x|$ then for $x \neq 0$,

$\frac{\sqrt{x}}{x} \leq K \Rightarrow \frac{1}{\sqrt{x}} \leq K \Rightarrow \sqrt{x} \geq \frac{1}{K} \Rightarrow x \geq \frac{1}{K^2} \Rightarrow x \in [\frac{1}{K^2}, 2]$

That is, $x \in (0, 2] \Rightarrow x \in [\frac{1}{K^2}, 2]$.

So $(0, 2] \subseteq [\frac{1}{K^2}, 2]$. Which is a contradiction since $K > 0$. Hence there is no number $K > 0$ such that $|g(x)| \leq K|x|$ for all $x \in I$. Or we can say that for $x, 0 \in I$ there exists no $K > 0$ such that $|g(x) - g(0)| \leq |x - 0|$ (Here $g(0) = 0$). Therefore, g **is not a Lipschitz function on** I.

3. The Uniform Continuity Theorem 2.3.2 and Theorem 2.3.4 can sometimes be combined to establish the uniform continuity of a function on a set.

We consider $g(x) := \sqrt{x}$ on the set $[0, \infty)$. Let $[0, \infty) = I \cup J$ where $I := [0, 2]$ and $J := [1, \infty)$. From theorem 2.3.2 g is continuous on I. For $x, u \in J$,

$|g(x) - g(u)| = |\sqrt{x} - \sqrt{u}| = \dfrac{|\sqrt{x} - \sqrt{u}|\,(\sqrt{x} + \sqrt{u})}{\sqrt{x} + \sqrt{u}} = \dfrac{|x - u|}{\sqrt{x} + \sqrt{u}} \leq \dfrac{1}{2}|x - u|$.

{ Since $x, u \in J$, $\sqrt{x}, \sqrt{u} \geq 1$. That is $\sqrt{x} + \sqrt{u} \geq 2$. So $\dfrac{1}{\sqrt{x} + \sqrt{u}} \leq \dfrac{1}{2}$ }. Thus g is Lipschitz function on J with constant $K = \dfrac{1}{2}$ and hence by Theorem 2.3.4, g is uniformly continuous on $[1, \infty)$.

g is uniformly continuous on $I \Rightarrow$ for any $\varepsilon > 0$ there exist $\delta_I(\varepsilon) > 0$ such that if $x, u \in I$ are any numbers satisfying $|x - u| < \delta_I(\varepsilon)$, then $|f(x) - f(u)| < \varepsilon$.
g is uniformly continuous on $J \Rightarrow$ for any $\varepsilon > 0$ there exist $\delta_J(\varepsilon) > 0$ such that if $x, u \in J$ are any numbers satisfying $|x - u| < \delta_J(\varepsilon)$, then $|f(x) - f(u)| < \varepsilon$.
Now choose $\delta(\varepsilon) = \min\{\delta_I(\varepsilon), \delta_J(\varepsilon)\}$.
$|x - u| < \delta(\varepsilon) \Rightarrow |x - u| < \delta_I(\varepsilon)$ and $|x - u| < \delta_J(\varepsilon)$. Hence for any $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ such that if $x, u \in I \cup J$ are any numbers satisfying $|x - u| < \delta(\varepsilon)$, then $|f(x) - f(u)| < \varepsilon$.
Therefore g is uniformly continuous on $[0, \infty)$.

The Continuous Extension Theorem

We have seen examples of functions that are continuous but not unifonnly continuous on open intervals; for example, the function $f(x) = 1/x$ on the interval $(0, 1)$. On the other hand by the Uniform Continuity Theorem, a function that is continuous on a closed bounded interval is always uniformly continuous. So the question arises: Under what conditions is a function uniformly continuous on a bounded open interval? The answer reveals the strength of uniform continuity, for it will be shown that a function on (a, b) is uniformly continuous if and only if it can be defined at the endpoints to produce a function that is continuous on the closed interval.

Theorem 2.3.6. *If $f : A \to \mathbb{R}$ is uniformly continuous on a subset A of $\mathbb{R}$ and if (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.*

Proof. Let (x_n) be a Cauchy sequence in A, and let $\varepsilon > 0$ be given. First choose $\delta > 0$ such that if x, u in A satisfy $|x - u| < \delta$, then $|f(x) - f(u)| < \varepsilon$. Since (x_n) is a Cauchy sequence, there exists $H(\delta)$ such that $|x_n - x_m| < \delta$ for all $n, m > H(\delta)$. By the choice of δ, this implies that for $n, m > H(\delta)$, we have $|f(x_n) - f(x_m)| < \varepsilon$. Therefore the sequence $(f(x_n))$ is a Cauchy sequence. $\square$

Note 2.3.7. We can show that the function $f(x) := 1/x$ is not uniformly continuous on $(0, 1)$ using Theorem 2.3.6. We note that the sequence given by $x_n := 1/n$ in $(0, 1)$ is a Cauchy sequence, but the image sequence, where $f(x_n) = n$, is not a Cauchy sequence. [You may have a doubt that $1/n \to 0$ in $\mathbb{R}$ but $0 \notin (0, 1)$ so $1/n$ is not a converging sequence in $(0, 1)$ then how a diverging sequence become cauchy sequence. Theorem 1.2.6 states the property of sequences in $\mathbb{R}$. this may not be true for any other subsets or super sets of $\mathbb{R}$. Here we are considering $(0, 1)$ instead of $\mathbb{R}$. Therefore we cannot use Theorem 1.2.6.]

Theorem 2.3.8. Continuous Extension Theorem *A function f is uniformly continuous on the interval (a, b) if and only if it can be defined at the endpoints a and b such that the extended function is continuous on $[a, b]$.*

Proof. Suppose f is uniformly continuous on (a, b). We shall show how to extend f to a; the argument for b is similar. This is done by showing that $\lim\limits_{x \to a} f(x) = L$ exists, and this is accomplished by using the sequential criterion for limits. If (x_n) is a sequence in (a, b) with $\lim(x_n) = a$, then it is a Cauchy sequence (since in $\mathbb{R}$ every converging sequence is a cauchy sequence.), and by the preceding theorem, the sequence $(f(x_n))$ is also a Cauchy sequence, and so is convergent by Theorem 1.2.6. Thus the limit $\lim(f(x_n)) = L$ exists. If (u_n) is any other sequence in (a, b)

that converges to a, then $\lim(u_n - x_n) = a - a = 0$, so by the uniform continuity of f we have

$$\begin{aligned}\lim(f(u_n)) &= \lim(f(u_n) - f(x_n)) + \lim(f(x_n)) \\ &= 0 + L = L.\end{aligned}$$

Since we get the same value L for every sequence converging to a, we infer from the sequential criterion for limits that f has limit L at a. If we define $f(a) := L$, then f is continuous at a. The same argument applies to b, so we conclude that f has a continuous extension to the interval $[a, b]$.
Conversely, assume that f can be defined at the endpoints a and b such that the extended function is continuous on $[a, b]$. The function f is continuous on $[a, b]$ implies that f is uniformly continuous on $[a, b]$. Consequently we can say that f is uniformly continuous on (a, b). □

Example 2.3.9. Let $f(x) := \sin(1/x)$ on $(0, b]$ for any real number $b > 0$. Then $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \sin(1/x)$ does not exists. Here f is continuous on $(0, b]$ but the function cannot be defined at 0 continuously. Hence by above theorem f is not uniformly continuous on $(0, b]$ for all $b > 0$.

Approximation

In many applications it is important to be able to approximate continuous functions by functions of an elementary nature. Although there are a variety of definitions that can be used to make the word "approximate" more precise, one of the most natural (as well as one of the most important) is to require that, at every point of the given domain, the approximating function shall not differ from the given function by more than the preassigned error.

Definition 19. A function $s : [a, b] \to \mathbb{R}$ is called a **step function** if $[a, b]$ is the union of a finite number of nonoverlapping intervals $I_1, I_2, I_3, \ldots, I_n$ such that s is constant on each interval, that is, $s(x) = c_k$ for all $x \in I_k$, $k = 1, 2, 3, 4, \ldots, n$.

Example 2.3.10. Consider the function $s : [-2, 4] \to \mathbb{R}$ defined by

$$s(x) := \begin{cases} 0, & -2 \leq x < -1, \\ 1, & -1 \leq x \leq 0, \\ \frac{1}{2} & \quad 0 < x < \frac{1}{2}, \\ 3, & \quad \frac{1}{2} \leq x < 1, \\ -2, & \quad 1 \leq x \leq 3, \\ 2, & \quad 3 < x \leq 4, \end{cases}$$

is a step function. (See Figure 2.7.)

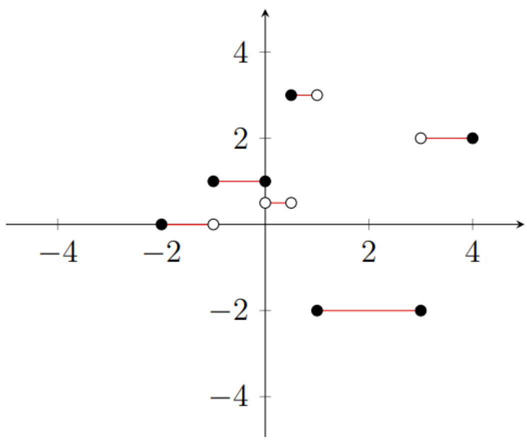

Figure 2.7:

Theorem 2.3.11. *Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. If $\varepsilon > 0$, then there exists a step functions $s_\varepsilon : I \to \mathbb{R}$ such that $|f(x) - S_\varepsilon(x)| < \varepsilon$ for all $x \in I$.*

Proof. Given that f is a continuous function on a closed and bounded interval then by Uniform Continuity Theorem 2.3.2 the function f is uniformly continuous, it follows that given $\varepsilon > 0$ there is a number $\delta(\varepsilon) > 0$ such that if $x, y \in I$ and $|x-y| < \delta(\varepsilon)$, then if $|f(x)-f(y)| < \varepsilon$. Let $I := [a, b]$ and let $m \in \mathbb{N}$ be sufficiently large so that $h := (b-a)/m < \delta(\varepsilon)$. We now divide $I = [a, b]$ into m disjoint intervals of length h; namely, $I_1 := [a, a+h]$, and $I_k := (a + (k-l)h, a+kh]$ for $k = 2, ..., m$. Since the length of each subinterval is $h < \delta(\varepsilon)$, the difference between any two values of f in I_k is less than ε. We now define

$$s_\varepsilon(x) := f(a+kh) \quad for\ x \in I_k,\ k = 1, ..., m, \tag{2.3}$$

So that s_ε is constant on each interval I_k. (In fact the value of f at the right endpoint of I_k see Figure 2.8) Consequently if $x \in I_k$, then

$$|f(x) - s_\varepsilon(x)| = |f(x) - f(a+kh)| < \varepsilon.$$

Therefore we have $|f(x) - s_\varepsilon(x)| < \varepsilon$ for all $x \in I$. □

Corollary 2.3.1. *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. If $\varepsilon > 0$, there exists a natural number m such that if we divide I into m disjoint intervals I_k having length $h := (b-a)/m$, then the step function s_ε defined in equation (1) satisfies $|f(x) - S_\varepsilon(x)| < \varepsilon$ for all $x \in I$.*

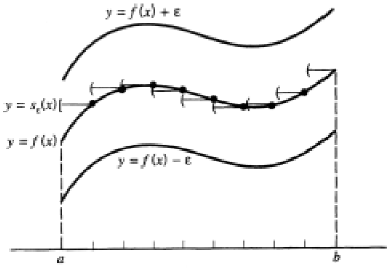

Figure 2.8: Approximation by step functions

Definition 20. Let $I := [a, b]$ be an interval. Then a function $g : I \to \mathbb{R}$ is said to be **piecewise linear** on I if I is the union of a finite number of disjoint intervals $I_1, ..., I_m$, such that the restriction of g to each interval I_k is a linear function.

Remark 2.3.2. It is evident that in order for a piecewise linear function g to be continuous on I, the line segments that form the graph of g must meet at the endpoints of adjacent subintervals I_k, I_{k+1} $(k = 1, ..., m-1)$.

Theorem 2.3.12. *Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. If $\varepsilon > 0$, then there exists a continuous piecewise linear function $g_\varepsilon : I \to \mathbb{R}$ such that $|f(x) - g_\varepsilon(x)| < \varepsilon$ for all $x \in I$.*

Proof. Since f is uniformly continuous on $I := [a, b]$, there is a number $\delta(\varepsilon) > 0$ such that if $x, y \in I$ and $|x - y| < \delta(\varepsilon)$, then $|f(x) - f(y)| < \varepsilon$. Let $m \in \mathbb{N}$ be sufficiently large so that $h := (b - a)/m < \delta(\varepsilon)$. Divide $I = [a, b]$ into m disjoint intervals of length h; namely, let $I_1 = [a, a + h]$, and let $I_k = (a + (k - 1)h, a + kh]$ for $k = 2, ..., m$. On each interval I_k we define g_ε to be the linear function joining the points $(a + (k - 1)h, f(a + (k - 1)h))$ and $(a + kh, f(a + kh))$.
So $g_\varepsilon(x) = f(a + (k-1)h) + \dfrac{f(a + kh) - f(a + (k-1)h)}{h}(x - (a + (k -$

$1)h))$ for $x \in I_k$.
Then g_ε is a continuous piecewise linear function on I. For $x \in I_k$, $|f(x) - f(a + (k-1)h)| < \varepsilon$ and $|f(x) - f(a + kh)| < \varepsilon$. Then for all $x \in I_k$,

$$
\begin{aligned}
|g_\varepsilon(x)| &= |f(a + (k-1)h) + \\
&\qquad \frac{f(a + kh) - f(a + (k-1)h)}{h}(x - (a + (k-1)h))\bigg| \\
&\leq |f(a + (k-1)h)| + \\
&\qquad \left|\frac{f(a + kh) - f(a + (k-1)h)}{h}\right| |(x - (a + (k-1)h))| \\
&< |f(a + (k-1)h)| + \left|\frac{f(a + kh) - f(a + (k-1)h)}{h}\right| |h| \\
&< |f(a + (k-1)h)| + |f(a + kh) - f(a + (k-1)h)|
\end{aligned}
$$

therefore[2] this inequality holds for all $x \in I$. (See Figure 2.9)

---

[2]Reason: If $x \in I_k$ then $|x - (a + (k-1)h)| < h < \delta(\varepsilon)$, so that $|f(x) - f(a + (k-1)h)| < \varepsilon$. Similarly, $|f(x) - f(a + kh)| < \varepsilon$.
We note that for $x \in I_k$,

$$
\begin{aligned}
g_\varepsilon(x) &= f(a + (k-1)h) \\
&\quad + \frac{f(a + kh) - f(a + (k-1)h)}{h} \cdot (x - (a + (k-1)h)).
\end{aligned}
$$

Hence for $x \in I_k$,

$$\begin{aligned}
|f(x) - g_\varepsilon(x)| &= \Big|f(x) - \Big\{f(a+(k-1)h) + \\
&\quad \frac{f(a+kh) - f(a+(k-1)h)}{h}(x - (a+(k-1)h))\Big\}\Big| \\
&\le |f(x) - f(a+(k-1)h)| + \\
&\quad \left|\frac{f(a+kh) - f(a+(k-1)h)}{h}\right| |(x - (a+(k-1)h))| \\
&< |f(x) - f(a+(k-1)h)| + \\
&\quad \left|\frac{f(a+kh) - f(a+(k-1)h)}{h}\right| |h| \\
&< |f(x) - f(a+(k-1)h)| + \\
&\quad |f(a+kh) - f(a+(k-1)h)|
\end{aligned}$$

□

Theorem 2.3.13. Weierstrass Approximation Theorem *Let $I = [a,b]$ and let $f : I \to \mathbb{R}$ be a continuous function. If $\delta > 0$ is given, then there exists a polynomial function P_ε such that $|f(x) - P_\varepsilon(x)| < \varepsilon$ for all $x \in I$.*

$$\begin{aligned}
|f(x) - g_\varepsilon(x)| &\le |f(x) - f(a+(k-1)h)| \\
&\quad + \frac{|f(a+kh) - f(a+(k-1)h)|}{h} \cdot h, \text{ as } |x - (a+(k-1)h)| < h \\
&\le |f(x) - f(a+(k-1)h)| + |f(a+kh) - f(a+(k-1)h)| \\
&\le |f(a+kh) - f(a+(k-1)h)| + |f(a+(k-1)h) - f(x)| \\
&\le |f(a) + kh) - f(x)|, \text{ by Triangle Inequality} \\
&< \varepsilon.
\end{aligned}$$

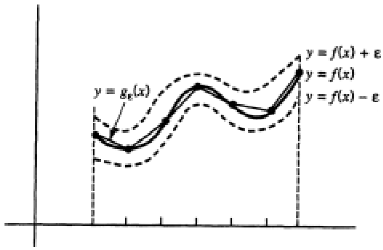

Figure 2.9: Approximation by piecewise linear function

EXERCISE

1. Show that the function $f(x) := 1/x$ is uniformly continuous on the set $A := [a, \infty)$, where a is a positive constant.

2. Show that the function $f(x) := 1/x^2$ is uniformly continuous on $A := [1, \infty)$, but that it is not unifonnly continuous on $B := (0, \infty)$.

3. Use the Nonuniform Continuity Criterion to show that the following functions are not uniformly continuous on the given sets.
   (a) $f(x) := x^2$, $A := [0, \infty)$.
   (b) $g(x) := \sin(1/x)$, $B := (0, \infty)$.

4. Show that the function $f(x) := 1/(1 + x^2)$ for $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

5. Show that if f and g are uniformly continuous on a subset A of $\mathbb{R}$, then $f + g$ is uniformly continuous on A.

6. Show that if f and g are uniformly continuous on $A \subseteq \mathbb{R}$ and if they are both bounded on A, then their product fg is uniformly continuous on A.

7. If $f(x) := x$ and $g(x) := \sin x$, show that both f and g are unifonnly continuous on $\mathbb{R}$, but that their product fg is not uniformly continuous on $\mathbb{R}$.

8. Prove that if f and g are each uniformly continuous on $\mathbb{R}$, then the composite function $f \circ g$ is uniformly continuous on $\mathbb{R}$.

9. If f is uniformly continuous on $A \subseteq \mathbb{R}$, and $|f(x)| \geq k > 0$ for all $x \in A$, show that $1/f$ is uniformly continuous on A.

10. Prove that if f is uniformly continuous on a bounded subset A of $\mathbb{R}$, then f is bounded on A.

11. If $g(x) := \sqrt{x}$ for $x \in [0, 1]$, show that there doesnot exist a constant K such that $|g(x)| \leq K|x|$ for all $x \in [0, 1]$. Conclude that the uniformly continuous g is not a Lipschitz function on $[0, 1]$. (Proved for $I = [0, 2]$.)

12. Show that if I is continuous on $[0, \infty)$ and uniformly continuous on $[a, \infty)$ for some positive constant a, then f is uniformly continuous on $[0, \infty)$.

13. Let $A \subseteq \mathbb{R}$ and suppose that $f : A \to \mathbb{R}$ has the following property: for each $\varepsilon > 0$ there exists a function $g_\varepsilon : A \to \mathbb{R}$ such that g_ε is uniformly continuous on A and $|f(x) - g_\varepsilon(x)| < \varepsilon$ for all $x \in A$. Prove that f is uniformly continuouson A.

14. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic on $\mathbb{R}$ if there exists a number $p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Prove that a continuous periodic function on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.

Chapter 3

RIEMANN INTEGRAL, RIEMANN INTEGRABLE FUNCTIONS AND THE FUNDAMENTAL THEOREM

In the 1660s Isaac Newton created his theory of "fluxions" and invented the method of "inverse tangents" to find areas under curves. The reversal of the process for finding tangent lines to find areas was also discovered in the 1680s by Gottfried Leibniz, who was unaware of Newton's unpublished work and who arrived at the discovery by a very different route. Leibniz introduced the terminology "calculus differentialis" and "calculus integralis", since finding tangent lines involved differences and finding areas involved summations. Thus, they had discovered that integration, being a process of summation, was inverse to the operation of differentiation. In the 1850s, Bernhard Riemann adopted a new and different viewpoint. He separated the concept of integration from its companion, differentiation, and examined the motivating summation and limit process of finding areas by itself. He broadened the scope by considering all functions on an interval for which this process of "integration" could be defined: the class of "integrable" functions. The Fundamental Theorem of Calculus became a result that held only for a restricted set of integrable functions.

3.1 Riemann Integral

We will define the Riemann integral as a kind of limit of the Riemann sums as the norm of the partitions tend to 0 and focus on the purely mathematical aspects of the integral.

Partitions and Tagged Partitions

Definition 21. If $I := [a, b]$ is a closed bounded interval in $\mathbb{R}$, then a **partition** of I is a finite, ordered set $\mathcal{P} := \{x_0, x_1, ..., x_{n-1}, x_n\}$ of points in I such that

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$$

Note 3.1.1. Let $\mathcal{P}$ be a partition of the interval $[a, b]$ (See Figure 3.1). The points of $\mathcal{P}$ are used to divide $I = [a, b]$ in to non-overlapping subintervals,

$$I_1 = [x_0, x_1],\ I_2 = [x_1, x_2],\ I_3 = [x_2, x_3], ...,\ I_n = [x_{n-1}, x_n].$$

often we will denote the partition $\mathcal{P}$ by the notation $\mathcal{P} = \{[x_{i-1}, x_i]\}$.

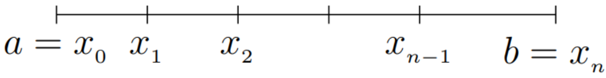

Figure 3.1:

Definition 22. Let $\mathcal{P} := \{x_0, x_1, ..., x_{n-1}, x_n\}$ be a partition of $[x_0, x_n]$ then we define **norm** (or **mesh**) of to be the number

$$||\mathcal{P}|| := \max\{x_1 - x_0, x_2 - x_1, ..., x_n - x_{n-1}\}.$$

Definition 23. Let $\mathcal{P} := \{x_0, x_1, ..., x_{n-1}, x_n\}$ be a partition of $[x_0, x_n] = I$. If a point t_i has been selected from each subinterval $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, ..., n$, then the points are called **tags** of the subintervals I_i. A set of ordered pairs

$$\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$$

of subintervals and corresponding tags is called a **tagged partition** of I (see Figure 3.2.).

Figure 3.2:

Note 3.1.2. The dot over the $\dot{\mathcal{P}}$ indicates that a tag has been chosen for each subinterval. The tags can be chosen in a wholly arbitrary fashion; for example, we can choose the tags to be the left endpoints, or the right endpoints, or the midpoints of the subintervals, etc. Note that an endpoint of a subinterval can be used as a tag for two consecutive subintervals. Since each tag can be chosen in infinitely many ways, each partition can be tagged in infinitely many ways. The norm of a tagged partition is defined as for an ordinary partition and does not depend on the choice of tags.

Definition 24. Let $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ is the tagged partition we define the **Riemann sum** of a function $f : [a, b] \to \mathbb{R}$ corresponding to $\dot{\mathcal{P}}$ to be the number

$$S(f; \dot{\mathcal{P}}) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

.

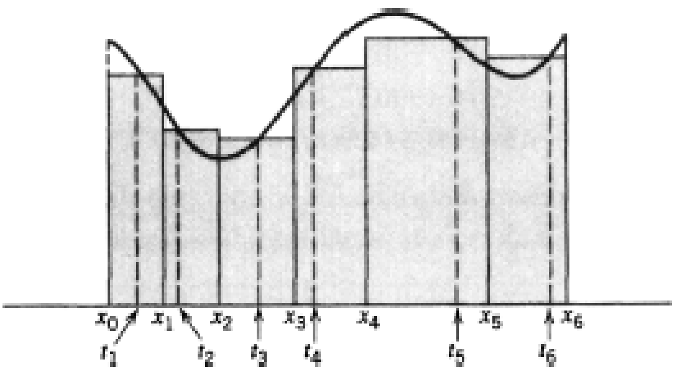

Figure 3.3: A Riemann sum

If the function f is positive on $[a, b]$, then the Riemann sum is the sum of the areas of n rectangles whose bases are the subintervals $I_i = [x_{i-1}, x_i]$ and whose heights are $f(t_i)$. (See Figure 3.3)

Definition of the Riemann Integral

We now define the Riemann integral of a function f on an interval $[a, b]$.

Definition 25. A function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** on $[a, b]$ if there exists a number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition of $[a, b]$ with $||\dot{\mathcal{P}}|| < \delta_\varepsilon$, then

$$|S(f; \dot{\mathcal{P}}) - L| < \varepsilon.$$

The set of all Riemann integrable functions on $[a, b]$ will be denoted by $\mathcal{R}[a, b]$.

Remark 3.1.1. It is sometimes said that the integral L is "the limit" of the Riemann sums $S(f; \dot{\mathcal{P}})$ as the norm $||\dot{\mathcal{P}}|| \to 0$. However, since $S(f; \dot{\mathcal{P}})$ is not a function of $||\dot{\mathcal{P}}||$, this limit is not of the type that we have studied before.

Theorem 3.1.3. *If $f \in \mathcal{R}[a, b]$, then the value of the integral is uniquely determined.*

Proof. Assume that $f \in \mathcal{R}[a, b]$. Show that the value of the integral is uniquely determined. Conversely assume that there exists L' and L'' which are the value of integral.
Then by definition of Riemann Integrable functions, for every $\varepsilon > 0$ there exists $\delta^{'}_{\varepsilon/2}, \delta^{''}_{\varepsilon/2} > 0$ such that if $\dot{\mathcal{P}}_1, \dot{\mathcal{P}}_2$ is any two tagged partition of $[a, b]$ with $||\dot{\mathcal{P}}_1|| < \delta^{'}_{\varepsilon/2}$ and $||\dot{\mathcal{P}}_2|| < \delta^{''}_{\varepsilon/2}$, then

$$|S(f; \dot{\mathcal{P}}_1) - L^{'}| < \varepsilon/2 \text{ and } |S(f; \dot{\mathcal{P}}_2) - L^{''}| < \varepsilon/2.$$

Now let $\delta_\varepsilon = \min\{\delta^{'}_{\varepsilon/2}, \delta^{''}_{\varepsilon/2}\}$ and let $\dot{\mathcal{P}}$ be a tagged partition with $||\dot{\mathcal{P}}|| < \delta_\varepsilon$. Since both $||\dot{\mathcal{P}}|| < \delta^{'}_{\varepsilon/2}$ and $||\dot{\mathcal{P}}|| < \delta^{''}_{\varepsilon/2}$, then

$$|S(f; \dot{\mathcal{P}}) - L^{'}| < \varepsilon/2 \text{ and } |S(f; \dot{\mathcal{P}}) - L^{''}| < \varepsilon/2.$$

Now consider $|L^{'} - L^{''}|$,

$$\begin{aligned}
|L^{'} - L^{''}| &= |L^{'} - L^{''} + S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{P}})| \\
&= |L^{'} - S(f;\dot{\mathcal{P}}) + S(f;\dot{\mathcal{P}}) - L^{''}| \\
&\leq |L^{'} - S(f;\dot{\mathcal{P}})| + |S(f;\dot{\mathcal{P}}) - L^{''}| \qquad \text{[By Triangle Inequality]} \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

So we get $|L^{'} - L^{''}| < \varepsilon$. Since $|L^{'} - L^{''}|$ is non negative and ε is any positive number, $|L^{'} - L^{''}|$ must be zero. That is, $L^{'} = L^{''}$. Therefore the value of the integral is unique. Hence we proved. □

Note 3.1.4. Theorem 3.1.3 shows that if $f \in \mathbb{R}[a, b]$, then the number L is uniquely determined. It will be called the **Riemann integral** of f over $[a, b]$. Instead of L, we will usually write

$$L = \int_a^b f \quad \text{or} \quad \int_a^b f(x)dx.$$

Theorem 3.1.5. *If g is Riemann integrable on $[a, b]$ and if $f(x) = g(x)$ except for a finite number of points in $[a, b]$, then f is riemann integrable and $\int_a^b f = \int_a^b g$.*

Proof. we prove this theorem by standard induction argument. Let n be the number of points where $f(x) \neq g(x)$ in the given interval.
Let c be a point in the interval and let $L = \int_a^b g$. Assume that $f(x) = g(x)$ for all $x \neq c$. For any tagged partition $\dot{\mathcal{P}}$, the terms in the two sums $S(f, \dot{\mathcal{P}})$ and $S(g, \dot{\mathcal{P}})$ are identical with the exception of at most two terms (in the case that $c = x_i$ or $c = x_{i-1}$ is an end point, an endpoint of a subinterval can be used as a tag for two consecutive subintervals). Let

$c = t_{i-1} = t_i$. Therefore, we have

$$\begin{aligned}
\left|S(f;\dot{\mathcal{P}}) - S(g;\dot{\mathcal{P}})\right| &= \left|\sum(f(t_i) - g(t_i))(x_i - x_{i-1})\right| \\
&= |(f(c) - g(c))(x_{i-1} - x_{i-2}) + \\
&\quad (f(c) - g(c))(x_i - x_{i-1})| \\
&\le |(f(c) - g(c))(x_{i-1} - x_{i-2})| + \\
&\quad |(f(c) - g(c))(x_i - x_{i-1})| \\
&\le |(f(c) - g(c))|\,|(x_{i-1} - x_{i-2})| + \\
&\quad |(f(c) - g(c))|\,|(x_i - x_{i-1})| \\
&< |(f(c) - g(c))|\,||\dot{\mathcal{P}}|| + |(f(c) - g(c))|\,||\dot{\mathcal{P}}|| \\
&< 2\,|(f(c) - g(c))|\,||\dot{\mathcal{P}}|| \\
&< 2(|f(c)| + |g(c)|)||\dot{\mathcal{P}}||. \qquad \text{[By Triangle Inequality]}
\end{aligned}$$

{ $2(|f(c)| + |g(c)|)||\dot{\mathcal{P}}|| < \varepsilon/2 \Rightarrow ||\dot{\mathcal{P}}|| < \varepsilon/(4(|f(c)| + |g(c)|))$. So when $||\dot{\mathcal{P}}|| < \delta_1 < \varepsilon/(4(|f(c)| + |g(c)|))$ then $\left|S(f;\dot{\mathcal{P}}) - S(g;\dot{\mathcal{P}})\right| < \varepsilon/2$ }
Now given $\varepsilon > 0$, we let $\delta_1 > 0$ satisfy $\delta_1 < \varepsilon/(4(|f(c))| + |g(c)|))$, and $\delta_2 > 0$ be such that $||\dot{\mathcal{P}}|| < \delta_2$ implies $|S(g;\dot{\mathcal{P}}) - L| < \varepsilon/2$. We now let $\delta := \min\{\delta_1, \delta_2\}$. Then, if $||\dot{\mathcal{P}}|| < \delta$, we obtain

$$\begin{aligned}
\left|S(f;\dot{\mathcal{P}}) - L\right| &= \left|S(f;\dot{\mathcal{P}}) - L + S(g;\dot{\mathcal{P}}) - S(g;\dot{\mathcal{P}})\right| \\
&= \left|\left[S(f;\dot{\mathcal{P}}) - S(g;\dot{\mathcal{P}})\right] + \left[S(g;\dot{\mathcal{P}}) - L\right]\right| \\
&\le \left|S(f;\dot{\mathcal{P}}) - S(g;\dot{\mathcal{P}})\right| + \left|S(g;\dot{\mathcal{P}}) - L\right| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Hence f is integrable with integral L.
Let $P(n)$: Let g is Riemann integrable on $[a, b]$ and if $f(x) = g(x)$ except for n number of points in $[a, b]$, then f is Riemann integrable and

$\int_a^b f = \int_a^b g$.

From above $P(n)$ is true for $n = 1$. Assume that $P(n)$ is true for $n = k$. Let g is Riemann integrable on $[a, b]$ and $f(x) = g(x)$ except for $k + 1$ number of points $c_1, c_2, ..., c_k, c_{k+1}$ in $[a, b]$.

$$\begin{aligned}\left|S(f;\dot{\mathcal{P}}) - S(g;\dot{\mathcal{P}})\right| &= \left|\sum(f(t_i) - g(t_i))(x_i - x_{i-1})\right| \\ &= |(f(c_1) - g(c_1))(x_{m_1} - x_{m_1-1}) + \\ &\quad (f(c_2) - g(c_2))(x_{m_2} - x_{m_2-1}) \\ &\quad + \cdots + (f(c_k) - g(c_k))(x_{m_k} - x_{m_k-1}) \\ &\quad +(f(c_{k+1}) - g(c_{k+1}))(x_{m_{k+1}} - x_{m_{k+1}-1})\big|\end{aligned}$$

Let $M = \max\{|(f(c_i) - g(c_i))| : i = 1, 2, ..., k, k + 1\}$. Then, we get $\left|S(f;\dot{\mathcal{P}}) - S(g;\dot{\mathcal{P}})\right| < 2(k + 1)M||\dot{\mathcal{P}}||$.

If we let $\delta_1 > 0$ satisfy $\delta_1 < \varepsilon/(4(k + 1)M)$ and $\delta_2 > 0$ be such that $||\dot{\mathcal{P}}|| < \delta_2$ implies $|S(g;\dot{\mathcal{P}}) - L| < \varepsilon/2$. We now let $\delta := \min\{\delta_1, \delta_2\}$. Then, if $||\dot{\mathcal{P}}|| < \delta$, we obtain

$\left|S(f;\dot{\mathcal{P}}) - L\right| < \varepsilon$. Threfore $P(k + 1)$ is also true. Hence we proved. □

Example 3.1.6.

(a) Every constant function on $[a, b]$ is in $\mathcal{R}[a, b]$.
Let $f(x) := k$ for all $x \in [a, b]$. If $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is any tagged partition of $[a, b]$, then it is clear that

$$S(f;\dot{\mathcal{P}}) = \sum_{i=1}^{n} k(x_i - x_{i-1}) = k(b - a).$$

Hence, for any $\varepsilon > 0$, we can choose $\delta_\varepsilon := 1$ so that if $||\dot{\mathcal{P}}|| < \delta_\varepsilon$, then

$$|S(f;\dot{\mathcal{P}}) - k(b-a)| = 0 < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f \in \mathcal{R}[a,b]$ and $\int_a^b f = k(b-a)$.

(b) Let $g : [0,3] \to \mathbb{R}$ be defined by $g(x) := 2$ for $0 \le x \le 1$, and $g(x) := 3$ for $1 < x \le 3$. A preliminary investigation, based on the graph of g (see Figure **??**), suggests that we might expect that $\int_0^3 g = 8$.

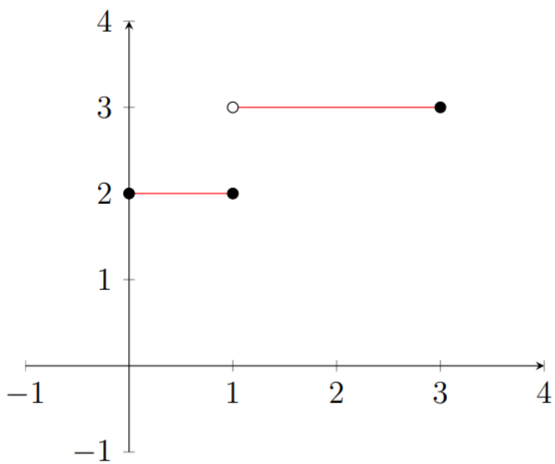

Figure 3.4:

Let $\dot{\mathcal{P}}$ be a tagged partition of $[0,3]$ with norm $< \delta$; we will show how to determine δ in order to ensure that $|S(g;\dot{\mathcal{P}}) - 8| < \varepsilon$. Let $\dot{\mathcal{P}}_1$ be the subset of $\dot{\mathcal{P}}$ having its tags in $[0,1]$ where $g(x) = 2$, and let $\dot{\mathcal{P}}_2$ be the subset of $\dot{\mathcal{P}}$ with its tags in $(1,3]$ where $g(x) = 3$. It is obvious that we have

$$S(g;\dot{\mathcal{P}}) = S(g;\dot{\mathcal{P}}_1) + S(g;\dot{\mathcal{P}}_2). \tag{3.1}$$

If we let U_1 denote the union of the subintervals in $\dot{\mathcal{P}}_1$, then it is readily shown that

$$[0, 1-\delta] \subset U_1 \subset [0, 1+\delta]. \tag{3.2}$$

To prove the first inclusion, we let $u \in [0, 1-\delta]$. Then u lies in an interval $I_k := [x_{k-1}, x_k]$ of $\dot{\mathcal{P}}_1$, and since $||\dot{\mathcal{P}}|| < \delta$, we have $x_k - x_{k-1} < \delta$. Then $x_{k-1} \leq u \leq 1-\delta$ implies that $x_k < x_{k-1} + \delta \leq (1-\delta) + \delta \leq 1$. Thus the tag t_k in I_k satisfies $t_k \leq 1$ and therefore u belongs to a subinterval whose tag is in $[0, 1]$, that is $u \in U_1$. This proves the first inclusion in (2), and the second inclusion can be shown in the same manner. Since $g(t_k) = 2$ for the tags of $\dot{\mathcal{P}}_1$ and since the interval in (2) have lengths $1-\delta$ and $1+\delta$, respectively, it follows that

$$2(1-\delta) \leq S(g;\dot{\mathcal{P}}_1) \leq 2(1+\delta).$$

A similar argument shows that the union of all subintervals with tag $t_i \in (1, 3]$ contains the interval $[1+\delta, 3]$ of length $2-\delta$, and is contained in $[1-\delta, 3]$ of length $2+\delta$.
Therefore,

$$3(2-\delta) \leq S(g;\dot{\mathcal{P}}_2) \leq 3(2+\delta).$$

adding these inequalities and using equation (1), we have

$$8 - 5\delta \leq S(g;\dot{\mathcal{P}}) = S(g;\dot{\mathcal{P}}_1) + S(g;\dot{\mathcal{P}}_2) \leq 8 + 5\delta,$$

whence it follows that

$$|S(g;\dot{\mathcal{P}}) - 8| \leq 5\delta$$

To have the final term $< \varepsilon$ we are led to take $\delta_\varepsilon < \varepsilon/5$
Making such a choice (for example, if we take $\delta_\varepsilon := \varepsilon/10$), we can retrace the argument and see that $|S(g;\dot{\mathcal{P}}) - 8| < \varepsilon$ when $||\dot{\mathcal{P}}|| < \delta_\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have proved that $g \in \mathcal{R}[0,3]$ and that $\int_0^3 g = 8$, as predicted.

(c) Let $h(x) := x$ for $x \in [0,1]$; we will show that $h \in \mathcal{R}[0,1]$. We will employ a "trick" that enables us to guess the value of the integral by considering a particular choice of the tag points. Indeed, if $\{I_i\}_{i=1}^n$ is any partition of $[0,1]$ and we choose the tag of the interval $I_i = [x_{i-1}, x_i]$ to be the midpoint $q_i := \frac{1}{2}(x_{i-1} + x_i)$ then the contribution of this term to the Riemann sum corresponding to the tagged partition $\dot{\mathcal{Q}} := \{(I_i, q_i)\}_{i=1}^n$ is

$$h(q_i)(x_i - x_{i-1}) = \frac{1}{2}(x_i + x_{i-1})(x_i) - x_{i-1} = \frac{1}{2}(x_i^2 - x_{i-1}^2).$$

If we add these terms and note that the sum telescopes, we obtain

$$S(h;\dot{\mathcal{Q}}) = \sum_{i=1}^{n} \frac{1}{2}(x_i^2 - x_{i-1}^2) = \frac{1}{2}(1^2 - 0^2) = \frac{1}{2}.$$

Now let $\dot{\mathcal{P}} := (I_i; t_i)_{i=1}^n$ be an arbitrary tagged partition of $[0,1]$ with $||\dot{\mathcal{P}}|| < \delta$ so that $x_i - x_{i-1} < \delta$ for $i = 1, ..., n$. Also let $\dot{\mathcal{Q}}$ have the same partition points, but where we choose the tag q_i to be the midpoint of the interval I_i. Since both t_i and q_i belong to this interval, we have $|t_i - q_i| < \delta$. Using the Triangle Inequality, we deduce

$$\begin{aligned}\left|S(f;\dot{\mathcal{P}})-S(h;\dot{\mathcal{Q}})\right| &= \left|\sum_{i=1}^{n} t_i(x_i-x_{i-1})-\sum_{i=1}^{n} q_i(x_i-x_{i-1})\right| \\ &\le \sum_{i=1}^{n} |t_i-q_i|\,(x_i-x_{i-1}) \\ &< \delta\sum_{i=1}^{n}(x_i-x_{i-1}) = \delta(x_n-x_0)=\delta.\end{aligned}$$

Since $S(h;\dot{\mathcal{Q}})=\dfrac{1}{2}$, we infer that if $\dot{\mathcal{P}}$ is any partition with $||\dot{\mathcal{P}}||<\delta$, then

$$\left|S(h;\dot{\mathcal{P}})-\frac{1}{2}\right|<\delta.$$

Therefore we led to take $\delta_\varepsilon \le \varepsilon$. If we choose $\delta_\varepsilon := \varepsilon$, we can retrace the argument to

conclude that $h\in\mathcal{R}[0,1]$ and $\int_0^1 h=\int_0^1 x dx=\dfrac{1}{2}$.

(d) Let $G(x):=1/n$ for $x=1/n$ $(n\in\mathbb{N})$, and $G(x):=0$ elsewhere on $[0,1]$.

Given $\varepsilon>0$, the set $E:=\{x : G(x)\ge\varepsilon\}$ is a finite set. (For example. if $\varepsilon=1/10$, then $E=\{1,1/2,1/3,...,1/10\}$). If n is the number of points in E, we allow for the possibility that a tag may be counted twice if it is an endpoint and let $\delta=\varepsilon/2n$. For a given tagged partition $\dot{\mathcal{P}}$ such that $||\dot{\mathcal{P}}||<\delta$, we let $\dot{\mathcal{P}}_0$ be the subset of $\dot{\mathcal{P}}$ with all tags outside of E and let $\dot{\mathcal{P}}_1$ be the subset of $\dot{\mathcal{P}}$ with one or more tags in E. Since $G(x)<\varepsilon$ for each x outside of E and $G(x)<1$ for all x in $[0,1]$, we get

$$0\le S(G;\dot{\mathcal{P}})=S(G;\dot{\mathcal{P}}_0)+S(G;\dot{\mathcal{P}}_1)<\varepsilon+(2n)\delta=2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $G(x)$ is Riemann integrable with integral equal to 0.

Some Properties of the Integral

Here we have some general theorems which is useful in determining the value of the integral and of δ_ε.

Theorem 3.1.7. *Suppose that f and g are in $\mathcal{R}[a,b]$. Then:*

(a) If $k \in \mathbb{R}$, the function kf is in $\mathcal{R}[a,b]$ and

$$\int_a^b kf = k\int_a^b f.$$

(b) The function $f+g$ is in $\mathcal{R}[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

(c) If $f(x) \leq g(x)$ for all $x \in [a,b]$, then

$$\int_a^b f \leq \int_a^b g.$$

Proof. If $\dot{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a tagged partition of $[a, b]$,

$$\begin{aligned} S(kf; \dot{\mathcal{P}}) &= \sum_{i=1}^n (kf)(t_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n k(f)(t_i)(x_i - x_{i-1}) \\ &= k \sum_{i=1}^n (f)(t_i)(x_i - x_{i-1}) \\ &= kS(f; \dot{\mathcal{P}}) \end{aligned}$$

$$\begin{aligned} S(f+g; \dot{\mathcal{P}}) &= \sum_{i=1}^n (f+g)(t_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n [f(t_i) + g(t_i)](x_i - x_{i-1}) \\ &= \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) + \sum_{i=1}^n g(t_i)(x_i - x_{i-1}) \\ &= S(f; \dot{\mathcal{P}}) + S(g; \dot{\mathcal{P}}) \end{aligned}$$

If $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\begin{aligned} S(f; \dot{\mathcal{P}}) &= \sum_{i=1}^n (f)(t_i)(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^n g(t_i)(x_i - x_{i-1}) \\ &\leq S(g; \dot{\mathcal{P}}) \end{aligned}$$

(a) If $k = 0$ then result is obvious. Now choose $0 \neq k \in \mathbb{R}$. Given $\varepsilon > 0$, we can use the argument in the proof of the Uniqueness Theorem

3.1.3 to construct a number $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition with $||\dot{\mathcal{P}}|| < \delta_\varepsilon$, then

$$\left| S(f;\dot{\mathcal{P}}) - \int_a^b f \right| < \varepsilon/|k| \tag{3.3}$$

$\left|S(kf;\dot{\mathcal{P}}) - k\int_a^b f\right| = \left|kS(f;\dot{\mathcal{P}}) - k\int_a^b f\right| = |k|\left|S(f;\dot{\mathcal{P}}) - \int_a^b f\right| < |k|(\varepsilon/|k|) = \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $f \in \mathcal{R}[a,b]$ and $\int_a^b kf = k\int_a^b f$.

(b) Given $\varepsilon > 0$, we can use the argument in the proof of the Uniqueness Theorem 3.1.3 to construct a number $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition with $||\dot{\mathcal{P}}|| < \delta_\varepsilon$, then both

$$\left| S(f;\dot{\mathcal{P}}) - \int_a^b f \right| < \varepsilon/2 \quad \text{and} \quad \left| S(g;\dot{\mathcal{P}}) - \int_a^b g \right| < \varepsilon/2. \tag{3.4}$$

Note that

$$\begin{aligned}\left| S(f+g;\dot{\mathcal{P}}) - \left(\int_a^b f + \int_a^b g\right)\right| &= \left| S(f;\dot{\mathcal{P}}) + S(g;\dot{\mathcal{P}}) - \int_a^b f - \int_a^b g \right| \\ &\le \left| S(f;\dot{\mathcal{P}}) - \int_a^b f \right| + \left| S(g;\dot{\mathcal{P}}) - \int_a^b g \right| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f+g \in \mathcal{R}[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

(c) From equation (4), we have

$$\int_a^b f - \varepsilon/2 < S(f;\dot{\mathcal{P}}) \text{ and } S(g;\dot{\mathcal{P}}) < \int_a^b g + \varepsilon/2.$$

If we use the fact that $S(f;\dot{\mathcal{P}}) \leq S(g;\dot{\mathcal{P}})$, we have

$$\int_a^b f \leq \int_a^b g + \varepsilon.$$

But, since $\varepsilon > 0$ is arbitrary, we conclude that $\int_a^b f \leq \int_a^b g$.

$\square$

Boundedness Theorem

We now show that an unbounded function cannot be Riemann integrable.

Theorem 3.1.8. *If $f \in \mathcal{R}[a,b]$, then f is bounded on $[a,b]$.*

Proof. Assume that f is an unbounded function in $\mathcal{R}[a,b]$ with integral L. Then there exists $\delta > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition of $[a,b]$ with $||\dot{\mathcal{P}}|| < \delta$, then we have $|S(f;\dot{\mathcal{P}}) - L| < 1$, which implies that

$$\left|S(f;\dot{\mathcal{P}})\right| < L + 1. \tag{3.5}$$

Now let $\mathcal{Q} = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of $[a,b]$ with $||\mathcal{Q}|| < \delta$. Since $|f|$ is not bounded on $[a,b]$, then there exists at least one subinterval in $\mathcal{Q}$, say $[x_{k-1}, x_k]$, on which $|f|$ is not bounded for, if $|f|$ is bounded on each subinterval $[x_{i-1}, x_i]$ by M_i, then it is bounded on $[a,b]$ by $\max\{M_1, M_2, ..., M_n\}$. We will now pick tags for $\mathcal{Q}$ that will provide a contradiction to (3.5). We tag $\mathcal{Q}$ by $t_i := x_i$ for $i \neq k$ and we pick $t_k \in [x_{k-1}, x_k]$ such that
$|f(t_k)(x_k - x_{k-1})| > |L| + 1 + \left|\sum_{i\neq k} f(t_i)(x_i - x_{i-1})\right|.$
$\Rightarrow |f(t_k)(x_k - x_{k-1})| - \left|\sum_{i\neq k} f(t_i)(x_i - x_{i-1})\right| > |L| + 1.$

$\left|f(t_k)(x_k - x_{k-1}) + \sum_{i\neq k} f(t_i)(x_i - x_{i-1})\right| \geq |f(t_k)(x_k - x_{k-1})| - \left|\sum_{i\neq k} f(t_i)(x_i - x_{i-1})\right|$
(by using Triangle inequality, $|A+B| \geq |A| - |B|$).

$\Rightarrow \left|f(t_k)(x_k - x_{k-1}) + \sum_{i\neq k} f(t_i)(x_i - x_{i-1})\right| > |L| + 1.$

ie, $|S(f;\dot{\mathcal{Q}})| = \left|\sum_{i=1}^{k} f(t_i)(x_i - x_{i-1})\right| > |L| + 1.$

So $|S(f;\dot{\mathcal{Q}})| > |L|+1$. Which is a contradiction to the equation (3.5). Therefore f must be bounded. Hence we proved. □

Example 3.1.9. Consider Thomae's function $h : [0,1] \to \mathbb{R}$, for $x \in [0,1]$

$$h(x) = \begin{cases} 0, & if\ x = 0\ or\ x\ is\ an\ irrational\ number \\ \frac{1}{n}, & if\ x = \frac{m}{n}\ where\ m, n \in \mathbb{N}\ and\ gcd(m,n) = 1 \end{cases}$$

We will now show that $h \in \mathcal{R}[0,1]$. For $\varepsilon > 0$, the set $E := \{x \in [0,1] : h(x) \geq \varepsilon/2\}$ is a finite set. (For example if $\varepsilon/2 = 1/5$, then there are eleven values of x such that $h(x) \geq 1/5$, namely $E = \{0, 1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5\}$.). We let n be the number of elements in E and take $\delta := \varepsilon/(4n)$. If $\dot{\mathcal{P}}$ is a given tagged partition such that $||\dot{\mathcal{P}}|| < \delta$, then we separate $\dot{\mathcal{P}}$ into two subsets. We let $\dot{\mathcal{P}}_1$ be the collection of tagged intervals in $\dot{\mathcal{P}}$ that they have their tags in E, and we let $\dot{\mathcal{P}}_2$ be the subset of tagged intervals in $\dot{\mathcal{P}}$ that have their tags elsewhere in $[0,1]$. Allowing for the possibility that a tag of $\dot{\mathcal{P}}_1$ may be an endpoint of adjacent intervals, we see that $\dot{\mathcal{P}}_1$ has at most $2n$ intervals and the total length of these intervals can be at most $2n\delta = \varepsilon/2$. Also, we have

$0 < h(t_i) \le 1$ for each tag t_i in $\dot{\mathcal{P}}_1$. Consequently, we have $S(h;\dot{\mathcal{P}}_1) \le 1\cdot 2n\delta \le \varepsilon/2$. For tags t_i in $\dot{\mathcal{P}}_2$ we have $h(t_i) < \varepsilon/2$ and the total length of the subintervals in $\dot{\mathcal{P}}_2$ is clearly less than 1, so that $S(h;\dot{\mathcal{P}}_2) < (\varepsilon/2)\cdot 1 = \varepsilon/2$. Therefore, combining these results, we get

$$0 \le S(h;\dot{\mathcal{P}}) = S(h;\dot{\mathcal{P}}_1) + S(h;\dot{\mathcal{P}}_2) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we infer that $h \in \mathcal{R}[0,1]$ with integral 0.

EXERCISES

1. If $I := [0,4]$, calculate the norms of the following partitions:

   (a) $\mathcal{P}_1 := (0,1,2,4)$, (b) $\mathcal{P}_2 := (0,2,3,4)$,

   (c) $\mathcal{P}_3 := (0,1,1.5,2,3.4,4)$, (d) $\mathcal{P}_4 := (0,0.5,2.5,3.5,4)$.

2. If $f(x) := x^2$ for $[0,4]$, calculate the following Riemann sums, where $\dot{\mathcal{P}}_i$ has the same partition points as in Exercise 1, and the tags are selected as indicated.

   (a) $\dot{\mathcal{P}}_1$ with the tags at the left endpoints of the subintervals.

   (b) $\dot{\mathcal{P}}_1$ with the tags at the right endpoints of the subintervals.

   (c) $\dot{\mathcal{P}}_2$ with the tags at the left endpoints of the subintervals.

   (d) $\dot{\mathcal{P}}_2$ with the tags at the right endpoints of the subintervals.

3. Show that $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ if and only if there exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition with norm $||\dot{\mathcal{P}}|| \le \delta_\varepsilon$, then $|S(f;\dot{\mathcal{P}}) - L| \le \varepsilon$.

4. (a) Let $f(x) := 2$ if $0 \le x < 1$ and $f(x) := 1$ if $1 \le x \le 2$. Show that $f \in \mathcal{R}[0,2]$ and evaluate its integral.

   (b) Let $h(x) := 2$ if $0 \le x < 1$, $h(1) := 3$ and $h(x) := 1$ if $1 < x \le 2$. Show that $h \in \mathcal{R}[0,2]$ and evaluate its integral.

5. Use Mathematical Induction and properties of integrals to show that if $f_1, f_2, f_3, ..., f_n$ are in $\mathcal{R}[a,b]$ and if $k_1, k_2, ..., k_n \in \mathbb{R}$ then the linear combination $f = \sum_{i=1}^{n} k_i f_i$ belongs to $\mathcal{R}[a,b]$ and $\int_a^b f = \sum_{i=1}^{n} k_i \int_a^b f_i$.

6. If $f \in \mathcal{R}[a,b]$ and $|f(x)| \le M$ for all $x \in [a,b]$, show that $|\int_a^b f| \le M(b-a)$.

7. If $f \in \mathcal{R}[a,b]$ and if $(\dot{\mathcal{P}}_n)$ is any sequence of tagged partitions of $[a,b]$ such that
$||\dot{\mathcal{P}}_n|| \to 0$, prove that $\int_a^b f = \lim_n S(f; \dot{\mathcal{P}}_n)$.

8. Let $g(x) := 0$ if $x \in [0,1]$ is rational and $g(x) := 1/x$ if $x \in [0,1]$ is irrational. Explain why $g \notin \mathcal{R}[0,1]$. However, show that there exists a sequence $(\dot{\mathcal{P}}_n)$ of tagged partitions of $[a,b]$ such that $||\dot{\mathcal{P}}_n|| \to 0$ and $\lim_n S(g; \dot{\mathcal{P}}_n)$ exists.

9. Suppose that f is bounded on $[a,b]$ and that there exists two sequences of tagged partitions of $[a,b]$ such that $||\dot{\mathcal{P}}_n|| \to 0$ and $||\dot{\mathcal{Q}}_n|| \to 0$, but such that $\lim_n S(f; \dot{\mathcal{P}}_n) \ne \lim_n S(f; \dot{\mathcal{Q}}_n)$. Show that f is not in $\mathcal{R}[a,b]$.

10. Consider the Dirichlet function, defined by $f(x) := 1$ for rational x in $[0,1]$ and $f(x) = 1$ for irrational x in $[0,1]$. Use the preceding exercise to show that f is not Riemann integrable on $[0,1]$.

11. Suppose that $c \le d$ are points in $[a,b]$. If $\phi : [a,b] \to \mathbb{R}$ satisfies $\phi(x) = \alpha > 0$ for $x \in [c,d]$ and $\phi(x) = 0$ elsewhere in $[a,b]$, prove

that $\phi \in \mathcal{R}[a,b]$ and that
$\int_a^b \phi = \alpha(d-c)$. [Hint: Given $\varepsilon > 0$ let $\delta_\varepsilon := \varepsilon/4\alpha$ and show that if $||\dot{\mathcal{P}}|| < \delta_\varepsilon$ then we have $\alpha(d-c-2\delta_\varepsilon) \le S(\phi;\dot{\mathcal{P}}) \le \alpha(d-c-2\delta_\varepsilon)$.]

12. Let $0 < a < b$, let $\mathcal{Q}(x) := x^2$ for $x \in [a,b]$ and let $\mathcal{P} := \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of $[a,b]$. For each i, let q_i be the positive square root of $\frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)$.

    (a) Show that q_i satisfies $0 \le x_{i-1} \le q_i \le x_i$.

    (b) Show that $\mathcal{Q}(q_i)(x_i - x_{i-1}) = \frac{1}{3}(x_i^3 - x_{i-1}^3)$.

    (c) If $\dot{\mathcal{Q}}$ is the tagged partition with the same subintervals as $\mathcal{P}$ and the tags q_i, show that $S(\mathcal{Q};\dot{\mathcal{Q}}) = \frac{1}{3}(b^3 - a^3)$.

    (d) Use the argument in Example 2.1.12 (c) to show that $\mathcal{Q} \in \mathcal{R}[a,b]$ and
    $$\int_a^b \mathcal{Q} = \int_a^b x^2 dx = \frac{1}{3}(b^3 - a^3).$$

13. If $f \in \mathcal{R}[a,b]$ and $c \in \mathbb{R}$, we define g on $[a+c, b+c]$ by $g(y) := f(y-c)$. Prove that $g \in \mathcal{R}[a+c, b+c]$ and that $\int_{a+c}^{b+c} g = \int_a^b f$. The function g is called the $c-$translate of f.

3.2 Riemann Integrable Functions

We begin with a proof of the important Cauchy Criterion. We will then prove the Squeeze Theorem, which will be used to establish the Riemann integrability of several classes of functions (step functions, continuous functions, and monotone functions). Finally we will establish the Additivity Theorem.

We have already noted that direct use of the definition requires that we know the value of the integral. The Cauchy Criterion removes this need, but at the cost of considering two Riemann sums, instead of just one.

Theorem 3.2.1. Cauchy Criterion *A function $f : [a,b] \to \mathbb{R}$ belongs to $\mathcal{R}[a,b]$ if and only if for every $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are any tagged partitions of $[a,b]$ with $||\dot{\mathcal{P}}|| < \eta_\varepsilon$ and $||\dot{\mathcal{Q}}|| < \eta_\varepsilon$, then*

$$|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| < \varepsilon.$$

Proof. ($\Rightarrow$) If $f \in \mathcal{R}[a,b]$ with integral L, let $\eta_\varepsilon := \delta_{\varepsilon/2} > 0$ be such that if $\dot{\mathcal{P}}, \dot{\mathcal{Q}}$ are tagged partitions such that $||\dot{\mathcal{P}}|| < \eta_\varepsilon$ and $||\dot{\mathcal{Q}}|| < \eta_\varepsilon$, then

$$|S(f;\dot{\mathcal{P}}) - L| < \varepsilon/2 \quad \text{and} \quad |S(f;\dot{\mathcal{Q}}) - L| < \varepsilon/2.$$

Therefore we have

$$\begin{aligned}|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| &\le |S(f;\dot{\mathcal{P}}) - L + L - S(f;\dot{\mathcal{Q}})| \\ &\le |S(f;\dot{\mathcal{P}}) - L| + |L - S(f;\dot{\mathcal{Q}})| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

($\Leftarrow$) For each $n \in \mathbb{N}$, let $\delta_n > 0$ be such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are tagged partitions with $||\dot{\mathcal{P}}|| < \delta_n$ and $||\dot{\mathcal{Q}}|| < \delta_n$, then

$$|S(f;\dot{\mathcal{P}}) - f(f;\dot{\mathcal{Q}})| < 1/n.$$

Evidently we may assume that $\delta_n \ge \delta_{n+1}$ for $n \in \mathbb{N}$; otherwise, we replace δ_n by $\delta_n' := \min\{\delta_1, \ldots, \delta_n\}$

For each $n \in \mathbb{N}$, let $\dot{\mathcal{P}}_n$ be a tagged partition with $||\dot{P}_n|| < \delta_n$. So if $m > n$, $||\dot{\mathcal{P}}_n|| < \delta_n$ and $||\mathcal{P}_m|| < \delta_m < \delta_{m-1} < .... < \delta_n$. That is, both tagged partition have norms $< \delta_n$, so that

$$\left|S\left(f; \dot{\mathcal{P}}_n\right) - S\left(f; \dot{\mathcal{P}}_m\right)\right| < 1/n \quad \text{for} \quad m > n \tag{3.6}$$

Consequently, the sequence $\left(S\left(f; \dot{\mathcal{P}}_m\right)\right)_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Therefore (by Theorem 1.2.6) this sequence converges in $\mathbb{R}$ and we let $A := \lim_m S\left(f; \dot{\mathcal{P}}_m\right)$.

Passing to the limit in (1) as $m \to \infty$, we have

$$\left|S\left(f; \dot{\mathcal{P}}_n\right) - A\right| \leq 1/n \quad \text{for all} \quad n \in \mathbb{N}$$

To see that A is the Riemann integral of f, given $\varepsilon > 0$, let $K \in \mathbb{N}$ satisfy $K > 2/\varepsilon$. If $\dot{Q}$ is any tagged partition with $||\dot{\mathcal{Q}}|| < \delta_K$, then

$$\begin{aligned}|S(f; \dot{\mathcal{Q}}) - A| &\leq \left|S(f; \dot{\mathcal{Q}}) - S\left(f; \dot{\mathcal{P}}_K\right)\right| + \left|S\left(f; \dot{\mathcal{P}}_K\right) - A\right| \\ &\leq 1/K + 1/K < \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then $f \in \mathcal{R}[a, b]$ with integral A.

□

Example 3.2.2.

(a) Let $g : [0, 3] \to \mathbb{R}$ be the function considered in Example 3.1.6(*b*). In that example we saw that if $\dot{\mathcal{P}}$ is a tagged partition of $[0, 3]$ with norm $||\dot{\mathcal{P}}|| < \delta$, then

$$8 - 5\delta \leq S(g;\dot{\mathcal{P}}) \leq 8 + 5\delta.$$

Hence if $\dot{\mathcal{Q}}$ is another tagged partition with $||\dot{\mathcal{Q}}|| < \delta$, then

$$8 - 5\delta \leq S(g;\dot{\mathcal{Q}}) \leq 8 + 5\delta$$

So $-(8-5\delta) \geq -S(g;\dot{\mathcal{Q}}) \geq -(8+5\delta) \Rightarrow -8-5\delta \leq -S(g;\dot{\mathcal{Q}}) \leq -8+5\delta$.

$$\begin{array}{lllll} 8-5\delta & \leq & S(g;\dot{\mathcal{Q}}) & \leq & 8+5\delta \quad + \\ -8-5\delta & \leq & -S(g;\dot{\mathcal{Q}}) & \leq & -8+5\delta \\ \hline \end{array}$$
$$-10\delta \leq S(g;\dot{\mathcal{Q}}) - S(g;\dot{\mathcal{Q}}) \leq 5\delta$$

ie,$|S(g;\dot{\mathcal{P}}) - S(g;\dot{\mathcal{Q}})| \leq 10\delta$.

In order to make this final term $< \varepsilon$, cosider $10\delta < \varepsilon \Rightarrow \delta < \varepsilon/10$. So we are led to employ the Cauchy Criterion with $\eta_\varepsilon := \varepsilon/20$.

(b) The Cauchy Criterion can be used to show that a function $f : [a,b] \to \mathbb{R}$ is not Riemann integrable. To do this we need to show that: There exists $\varepsilon_0 > 0$ such that for any $\eta > 0$ there exists tagged partitions $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ with $||\dot{\mathcal{P}}|| < \eta$ and $|\dot{\mathcal{Q}}|| < \eta$ such that $|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| \geq \varepsilon_0$

We will apply these remarks to the Dirichlet function, defined by $f(x) := 1$ if $x \in [0,1]$ is rational and $f(x) := 0$ if $x \in [0,1]$ is irrational.

Here we take $\varepsilon_0 := \frac{1}{2}$. If $\dot{\mathcal{P}}$ is any partition all of whose tags are rational numbers then $S(f;\dot{\mathcal{P}}) = 1$, while if $\dot{\mathcal{Q}}$ is any tagged partition all of whose tags are irrational numbers then $S(f;\dot{\mathcal{Q}}) = 0$. Since we are able to take such tagged partitions with arbitrarily small norms, we conclude that the Dirichlet function is not Riemann integrable.

The Squeeze Theorem

In working with the definition of Riemann integral, we have encountered two types of difficulties. First, for each partition, there are infinitely many choices of tags. And second, there are infinitely many partitions that have a norm less than a specified amount. We have experienced dealing with these difficulties in examples and proofs of theorems. We will now establish an important tool for proving integrability called the Squeeze Theorem that will provide some relief from those difficulties. It states that if a given function can be "squeezed" or bracketed between two functions that are known to be Riemann integrable with sufficient accuracy, then we may conclude that the given function is also Riemann integrable. The exact conditions are given in the statement of the theorem.

Theorem 3.2.3. Squeeze Theorem *Let $f : [a, b] \to \mathbb{R}$. Then $f \in \mathcal{R}[a, b]$ if and only if for every $\varepsilon > 0$ there exist functions α_ε and ω_ε in $\mathcal{R}[a, b]$ with*

$$\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x) \quad \textit{for all} \quad x \in [a, b], \tag{3.7}$$

and such that

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon. \tag{3.8}$$

Proof. ($\Rightarrow$) Take $\alpha_\varepsilon = \omega_\varepsilon = f$ for all $\varepsilon > 0$. Then $\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x)$ for all $x \in [a,b]$ and $\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) = 0 < \varepsilon$.

($\Leftarrow$) Let $\varepsilon > 0$. Since α_ε and ω_ε belong to $\mathcal{R}[a,b]$, there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition with $||\dot{\mathcal{P}}|| < \delta_\varepsilon$ then

$$\left| S\left(\alpha_\varepsilon; \dot{\mathcal{P}}\right) - \int_a^b \alpha_\varepsilon \right| < \varepsilon \quad \text{and} \quad \left| S\left(\omega_\varepsilon; \dot{\mathcal{P}}\right) - \int_a^b \omega_\varepsilon \right| < \varepsilon.$$

$$\Rightarrow -\varepsilon < S\left(\alpha_\varepsilon; \dot{\mathcal{P}}\right) - \int_a^b \alpha_\varepsilon < \varepsilon \quad \text{and} \; -\varepsilon < S\left(\omega_\varepsilon; \dot{\mathcal{P}}\right) - \int_a^b \omega_\varepsilon < \varepsilon$$

It follows from these inequalities that

$$\int_a^b \alpha_\varepsilon - \varepsilon < S\left(\alpha_\varepsilon; \dot{\mathcal{P}}\right) \quad \text{and} \quad S\left(\omega_\varepsilon; \dot{\mathcal{P}}\right) < \int_a^b \omega_\varepsilon + \varepsilon.$$

In view of inequality (2), we have $S\left(\alpha_\varepsilon; \dot{\mathcal{P}}\right) \le S(f; \dot{\mathcal{P}}) \le S\left(\omega_\varepsilon; \dot{\mathcal{P}}\right)$, whence

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(f; \dot{\mathcal{P}}) < \int_a^b \omega_\varepsilon + \varepsilon.$$

If $\mathcal{Q}$ is another tagged partition with $\|\mathcal{Q}\| < \delta_\varepsilon$, then we also have

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(f; \dot{\mathcal{Q}}) < \int_a^b \omega_\varepsilon + \varepsilon.$$

If we subtract these two inequalities and use (3), we conclude that

$$\begin{aligned} |S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| &< \int_a^b \omega_\varepsilon - \int_a^b \alpha_\varepsilon + 2\varepsilon \\ &= \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) + 2\varepsilon < 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the Cauchy Criterion implies that $f \in \mathcal{R}[a,b]$.

□

Classes of Riemann Integrable Functions

The Squeeze Theorem is often used in connection with the class of step functions. A function $\varphi : [a,b] \to \mathbb{R}$ is a step function if it has only a finite number of distinct values, each value being assumed on one or more subintervals of $[a,b]$.

Lemma 3.2.1. *If J is a subinterval of $[a,b]$ having endpoints $c < d$ and if $\varphi_J(x) := 1$ for $x \in J$ and $\varphi_J(x) := 0$ elsewhere in $[a,b]$, then $\varphi_J \in \mathcal{R}[a,b]$ and $\int_a^b \varphi_J = d - c$.*

Proof. Let $J = [c,d]$ with $c < d$ and $\dot{\mathcal{P}}$ be a tagged partition of $[a,b]$ with norm less than δ. Let $\dot{\mathcal{P}}_1$ be the subset of $\dot{\mathcal{P}}$ having its tags in $[a,c)$ where $\varphi_J(x) := 0$. Let $\dot{\mathcal{P}}_2$ be the subset of $\dot{\mathcal{P}}$ having its tags in $[c,d]$ where $\varphi_J(x) := 1$ and let $\dot{\mathcal{P}}_3$ be the subset of $\dot{\mathcal{P}}$ having its tags in $(d,b]$ where $\varphi_J(x) := 0$. It is obvious that we have

$$S(\varphi_J;\dot{\mathcal{P}}) = S(\varphi_J;\dot{\mathcal{P}}_1) + S(\varphi_J;\dot{\mathcal{P}}_2) + S(\varphi_J;\dot{\mathcal{P}}_3). \tag{3.9}$$

If we let U_1 denote the union of the subintervals in $\dot{\mathcal{P}}_1$, U_2 denote the union of the subintervals in $\dot{\mathcal{P}}_2$ and U_3 denote the union of the subintervals in $\dot{\mathcal{P}}_3$, then

$$[a, c-\delta] \subset U_1 \subset [a, c+\delta], \quad [c+\delta, d-\delta] \subset U_2 \subset [c-\delta, d+\delta] \quad \text{and}$$

$$[d+\delta, b] \subset U_3 \subset [d-\delta, b].$$

Hence it follows that $0.(c-\delta-a) \leq S(\varphi_J; \dot{\mathcal{P}}_1) \leq 0.(c+\delta-a)$, $1.(d-\delta-(c+\delta)) \leq S(\varphi_J; \dot{\mathcal{P}}_2) \leq 1.(d+\delta-(c-\delta))$ and $0.(b-(d+\delta)) \leq S(\varphi_J; \dot{\mathcal{P}}_3) \leq 0.(b-(d-\delta))$.
ie, $0 \leq S(\varphi_J; \dot{\mathcal{P}}_1) \leq 0$, $d-c-2\delta \leq S(\varphi_J; \dot{\mathcal{P}}_2) \leq d-c+2\delta$ and $0 \leq S(\varphi_J; \dot{\mathcal{P}}_3) \leq 0$.

By adding these three equations we get, $d-c-2\delta \leq S(\varphi_J; \dot{\mathcal{P}}_1) + S(\varphi_J; \dot{\mathcal{P}}_2) + S(\varphi_J; \dot{\mathcal{P}}_3) \leq d-c+2\delta$.
So $d-c-2\delta \leq S(\varphi_J; \dot{\mathcal{P}}) \leq d-c+2\delta$. Hence

$$|S(\varphi_J; \dot{\mathcal{P}}) - (d-c)| \leq 2\delta$$

To have this final term $< \varepsilon$ we are led to take $\delta_\varepsilon < \varepsilon/2$ and we can choose $\delta_\varepsilon := \varepsilon/4$. Then $\int_a^b \varphi_J = d-c$.
There are three other subintervals J having the same endpoints c and d, namely, $[c, d)$, $(c, d]$, and (c, d). Since, by Theorem 3.1.5, we can change the value of a function at finitely many points without changing the integral, we have the same result for these other three subintervals. Therefore, we conclude that all four functions φ_J are integrable with integral equal to $d-c$. □

Theorem 3.2.4. *If $\varphi : [a, b] \to \mathbb{R}$ is a step function, then $\varphi \in \mathcal{R}[a, b]$.*

Proof. Step functions of the type appearing in Lemma 3.2.1 are called "elementary step functions". An arbitrary step function φ can be expressed as a linear combination of such elementary step functions:

$$\varphi = \sum_{j=1}^{m} k_j \varphi_{J_j}$$

where J_j has endpoints $c_j < d_j$. From Lemma 3.2.1, it is clear that, $\varphi_{J_1}, \varphi_{J_2}, ..., \varphi_{J_m}$ are in $\mathcal{R}[a,b]$. Using properties of Riemann integrable functions we can say that

$\sum_{j=1}^{m} k_j \varphi_{J_j} \in \mathcal{R}[a,b]$. Hence $\varphi \in \mathcal{R}[a,b]$ and that

$$\int_a^b \varphi = \sum_{j=1}^{m} k_j \left(d_j - c_j\right)$$

□

Note 3.2.5. Any step function is Riemann integrable.

Example 3.2.6.

(a) The function g defined by $g(x) = 2$ for $0 \leq x \leq 1$ and $g(x) = 3$ for $1 < x \leq 3$. We now see that g is a step function and therefore we calculate its integral to be
$\int_0^3 g = 2 \cdot (1-0) + 3 \cdot (3-1) = 2 + 6 = 8.$

(b) Let $h(x) := x$ on $[0,1]$ and let $P_n := (0, 1/n, 2/n, \ldots, (n-1)/n, n/n = 1)$. We define the step functions α_n and ω_n on the disjoint subintervals $[0, 1/n), [1/n, 2/n), \ldots, [(n-2)/n), (n-1)/n), [(n-1)/n, 1]$ as follows:

$\alpha_n(x) := h((k-1)/n) = (k-1)/n$ for x in $[(k-1)/n, k/n)$ for $k = 1, 2, \ldots, n-1$, and
$\alpha_n(x) := h((n-1)/n) = (n-1)/n$ for x in $[(n-1)/n, 1]$.

That is, α_n has the minimum value of h on each subinterval. Similarly, we define ω_n to be the maximum value of h on each subinterval, that is,

$\omega_n(x) := k/n$ for x in $[(k-1)/n, k/n)$ for $k = 1, 2, \ldots, n-1$, and
$\omega_n(x) := 1$ for x in $[(n-1)/n, 1]$.

Then we get

$$\begin{aligned}
\int_0^1 \alpha_n &= \frac{1}{n}(0 + 1/n + 2/n + \cdots + (n-1)/n) \\
&= \frac{1}{n^2}(1 + 2 + \cdots + (n-1)) \\
&= \frac{1}{n^2}\frac{(n-1)n}{2} = \frac{1}{2}(1 - 1/n)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \omega_n &= \frac{1}{n}(1/n + 2/n + \cdots + (n-1)/n + n/n) \\
&= \frac{1}{n^2}(1 + 2 + \cdots + n) \\
&= \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2}(1 + 1/n)
\end{aligned}$$

Thus we have $\alpha_n(x) \leq h(x) \leq \omega_n(x)$ for $x \in [0, 1]$ and

$$\int_0^1 (\omega_n - \alpha_n) = \frac{1}{n}.$$

Since for a given $\varepsilon > 0$, we can choose n so that $\frac{1}{n} < \varepsilon$, it follows from the Squeeze Theorem that h is integrable. We also see that the value of the integral of h lies between the integrals of α_n and ω_n for all n and therefore has value $\frac{1}{2}$.

Theorem 3.2.7. *$f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then $f \in \mathcal{R}[a, b]$.*

Proof. Using the fact that a real valued continuous function on a closed and bounded interval uniformly continuous we get f is uniformly continuous on $[a, b]$. Therefore, given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $u, v \in [a, b]$ and $|u - v| < \delta_\varepsilon$, then we have $|f(u) - f(v)| < \varepsilon/(b - a)$.

Let $\mathcal{P} = \{I_i\}_{i=1}^{n}$ be a partition such that $\|\mathcal{P}\| < \delta_\varepsilon$. Applying Maximum-Minimum Theorem 2.2.7 we let $u_i \in I_i$ be a point where f attains its minimum value on I_i, and let $v_i \in I_i$ be a point where f attains its maximum value on I_i.
Let α_ε be the step function defined by
$\alpha_\varepsilon(x) := f(u_i)$ for $x \in [x_{i-1}, x_i)\, (i = 1, \ldots, n - 1)$ and
$\alpha_\varepsilon(x) := f(u_n)$ for $x \in [x_{n-1}, x_n]$.
Let ω_ε be defined similarly using the points v_i instead of the u_i.
$\omega_\varepsilon(x) := f(v_i)$ for $x \in [x_{i-1}, x_i)\, (i = 1, \ldots, n - 1)$ and
$\omega_\varepsilon(x) := f(v_n)$ for $x \in [x_{n-1}, x_n]$.
Then one has

$$\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x) \quad \text{for all} \quad x \in [a, b].$$

Moreover, it is clear that

$$\begin{aligned} 0 \le \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) &= \sum_{i=1}^{n} (f(v_i) - f(u_i))\,(x_i - x_{i-1}) \\ &< \sum_{i=1}^{n} \left(\frac{\varepsilon}{b - a}\right)(x_i - x_{i-1}) \end{aligned}$$

$$= \left(\frac{\varepsilon}{b-a} \right) \sum_{i=1}^{n} (x_i - x_{i-1}) = \varepsilon$$

Therefore it follows from the Squeeze Theorem that $f \in \mathcal{R}[a,b]$. $\square$

Theorem 3.2.8. *If $f : [a,b] \to \mathbb{R}$ is monotone on $[a,b]$, then $f \in \mathcal{R}[a,b]$.*

Proof. Assume that f is increasing on $I = [a,b]$. Partitioning the interval into n equal subintervals $I_k = [x_{k-1}, x_k]$ gives us $x_k - x_{k-1} = (b-a)/n, k = 1, 2, \ldots, n$. Since f is increasing on I_k, its minimum value is attained at the left endpoint x_{k-1} and its maximum value is attained at the right endpoint x_k. Therefore, we define the step functions
$\alpha(x) := f(x_{k-1})$ and $\omega(x) := f(x_k)$ for $x \in [x_{k-1}, x_k), k = 1, 2, \ldots, n-1$, and
$\alpha(x) := f(x_{n-1})$ and $\omega(x) := f(x_n)$ for $x \in [x_{n-1}, x_n]$.
Then we have $\alpha(x) \le f(x) \le \omega(x)$ for all $x \in I$, and

$$\int_a^b \alpha = \frac{b-a}{n} \left(f(x_0) + f(x_1) + \cdots + f(x_{n-1}) \right)$$
$$\int_a^b \omega = \frac{b-a}{n} \left(f(x_1) + \cdots + f(x_{n-1}) + f(x_n) \right).$$

Subtracting, and noting the many cancellations, we obtain

$$\int_a^b (\omega - \alpha) = \frac{b-a}{n} \left(f(x_n) - f(x_0) \right) = \frac{b-a}{n} (f(b) - f(a)).$$

Thus for a given $\varepsilon > 0$, we choose n such that $n > (b-a)(f(b) - f(a))/\varepsilon$. Then we have $\int_a^b (\omega - \alpha) < \varepsilon$ and the Squeeze Theorem implies that f is integrable on I. $\square$

Remark 3.2.2. Let $f : [a, b] \to \mathbb{R}$, then

- Any step function, f, is Riemann integrable on $[a, b]$.
- Any continuous function, f, is Riemann integrable on $[a, b]$.
- Any monotone function, f, is Riemann integrable on $[a, b]$.

The Additivity Theorem

We now return to arbitrary Riemann integrable functions. Our next result shows that the integral is an "additive function" of the interval over which the function is integrated.

Theorem 3.2.9. Additivity Theorem *Let $f := [a, b] \to \mathbb{R}$ and let $c \in (a, b)$. Then $f \in \mathcal{R}[a, b]$ if and only if its restrictions to $[a, c]$ and $[c, b]$ are both Riemann integrable. In this case*

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{3.10}$$

Proof. ($\Leftarrow$) Suppose that the restriction f_1 of f to $[a, c]$, and the restriction f_2 of f to $[c, b]$ are Riemann integrable to L_1 and L_2 respectively. Then, given $\varepsilon > 0$ there exists $\delta' > 0$ such that if $\dot{\mathcal{P}}_1$ is a tagged partition of $[a, c]$ with $||\dot{\mathcal{P}}_1|| < \delta'$, then $\left|S\left(f_1; \dot{\mathcal{P}}_1\right) - L_1\right| < \varepsilon/3$. Also there exists $\delta'' > 0$ such that if $\dot{\mathcal{P}}_2$ is a tagged partition of $[c, b]$ wilh $||\dot{\mathcal{P}}_2|| < \delta''$ then $\left|S\left(f_2; \dot{\mathcal{P}}_2\right) - L_2\right| < \varepsilon/3$. If M is a bound for $|f|$, we define $\delta_\varepsilon := \min\{\delta', \delta'', \varepsilon/6M\}$ and let $\dot{\mathcal{P}}$ be a tagged partition of $[a, b]$ with $\|\dot{\mathcal{Q}}\| < \delta$. We will prove that

$$\left|S(f;\dot{\mathcal{Q}}) - (L_1 + L_2)\right| < \varepsilon. \tag{3.11}$$

(i) If c is a partition point of $\dot{\mathcal{Q}}$, we split $\dot{\mathcal{Q}}$ into a partition $\dot{\mathcal{Q}}_1$ of $[a,c]$ and a partition $\dot{\mathcal{Q}}_2$ of $[c,b]$. Since $S(f;\dot{\mathcal{Q}}) = S\left(f;\dot{\mathcal{Q}}_1\right) + S\left(f;\dot{\mathcal{Q}}_2\right)$, and since $\dot{\mathcal{Q}}_1$ has norm $< \delta'$ and $\dot{\mathcal{Q}}_2$ has norm $< \delta''$, the inequality (3.11) is clear.

(ii) If c is not a partition point in $\dot{\mathcal{Q}} = \{(I_k, t_k)\}_{k=1}^{m}$, there exists $k \leq m$ such that $c \in (x_{k-1}, x_k)$. We let $\dot{\mathcal{Q}}_1$ be the tagged partition of $[a,c]$ defined by

$$\dot{\mathcal{Q}}_1 := \{(I_1, t_1), \ldots, (I_{k-1}, t_{k-1}), ([x_{k-1}, c], c)\}$$

and $\dot{\mathcal{Q}}_2$ be the tagged partition of $[c,b]$ defined by

$$\dot{\mathcal{Q}}_2 := \{([c, x_k], c), (I_{k+1}, t_{k+1}), \ldots, (I_m, t_m)\}$$

A straightforward calculation shows that

$$\begin{aligned}
S(f;\dot{\mathcal{Q}}) - S\left(f;\dot{\mathcal{Q}}_1\right) - S\left(f;\dot{\mathcal{Q}}_2\right) &= f(t_k)(x_k - x_{k-1}) - \\
&\quad f(c)(c - x_{k-1}) - f(c)(x_k - c) \\
&= f(t_k)(x_k - x_{k-1}) - \\
&\quad f(c)(x_k - x_{k-1}) \\
&= (f(t_k) - f(c)) \cdot (x_k - x_{k-1}),
\end{aligned}$$

whence it follows that

$$\begin{aligned}\left|S(f;\dot{\mathcal{Q}})-S\left(f;\dot{\mathcal{Q}}_1\right)-S\left(f;\dot{\mathcal{Q}}_2\right)\right|&=\left|\left(f\left(t_k\right)-f(c)\right)\cdot\left(x_k-x_{k-1}\right)\right|\\&\leq\left|\left(f\left(t_k\right)-f(c)\right)\right|\cdot\left|\left(x_k-x_{k-1}\right)\right|\\&\leq 2M\left(x_k-x_{k-1}\right)<\varepsilon/3\end{aligned}$$

But since $\|\mathcal{Q}_1\|<\delta\leq\delta'$ and $\|\mathcal{Q}_2\|<\delta\leq\delta''$, it follows that

$$\left|S\left(f;\dot{\mathcal{Q}}_1\right)-L_1\right|<\varepsilon/3\quad\text{and}\quad\left|S\left(f;\dot{\mathcal{Q}}_2\right)-L_2\right|<\varepsilon/3$$

from which we obtain (3.11). Since $\varepsilon>0$ is arbitrary, we infer that $f\in\mathcal{R}[a,b]$ and that (3.10) holds.

($\Rightarrow$) We suppose that $f\in\mathcal{R}[a,b]$ and, given $\varepsilon>0$, we let $\eta_\varepsilon>0$ satisfy the Cauchy Criterion 3.2.1. Let f_1 be the restriction of f to $[a,c]$ and let $\dot{\mathcal{P}}_1,\dot{\mathcal{Q}}_1$ be tagged partitions of $[a,c]$ with $\left\|\dot{\mathcal{P}}_1\right\|<\eta_\varepsilon$ and $\left\|\dot{\mathcal{Q}}_1\right\|<\eta_\varepsilon$. By adding additional partition points and tags from $[c,b]$, we can extend $\dot{\mathcal{P}}_1$ and $\dot{\mathcal{Q}}_1$ to tagged partitions $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ of $[a,b]$ that satisfy $\|\dot{\mathcal{P}}\|<\eta_\varepsilon$ and $\|\dot{\mathcal{Q}}\|<\eta_\varepsilon$. If we use the same additional points and tags in $[c,b]$ for both $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$, then

$$S\left(f_1;\dot{\mathcal{P}}_1\right)-S\left(f_1;\dot{\mathcal{Q}}_1\right)=S(f;\dot{\mathcal{P}})-S(f;\dot{\mathcal{Q}}).$$

Since both $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ have norm η_ε, then $\left|S\left(f_1;\dot{\mathcal{P}}_1\right)-S\left(f_1;\dot{\mathcal{Q}}_1\right)\right|<\varepsilon$. Therefore the Cauchy Condition shows that the restriction f_1 of f to $[a,c]$ is in $\mathcal{R}[a,c]$. In the same way, we see that the restriction f_2 of f to $[c,b]$ is in $\mathcal{R}[c,d]$.

The equality (3.10) now follows from the first part of the theorem.

□

Corollary 3.2.3. *If $f \in \mathcal{R}[a,b]$, and if $[c,d] \subseteq [a,b]$, then the restriction of f to $[c,d]$ is in $\mathcal{R}[c,d]$.*

Proof. Since $f \in \mathcal{R}[a,b]$ and $c \in [a,b]$, it follows from the theorem that its restriction to $[c,b]$ is in $\mathcal{R}[c,b]$. But if $d \in [c,b]$, then another application of the theorem shows that the restriction of f to $[c,d]$ is in $\mathcal{R}[c,d]$. □

Corollary 3.2.4. *If $f \in \mathcal{R}[a,b]$ and if $a = c_0 < c_1 < \cdots < c_m = b$, then the restrictions of f to each of the subintervals $[c_{i-1}, c_i]$ are Riemann integrable and*

$$\int_a^b f = \sum_{i=1}^{m} \int_{c_{i-1}}^{c_i} f$$

Definition 26. If $f \in \mathcal{R}[a,b]$ and if $\alpha, \beta \in [a,b]$ with $\alpha < \beta$, we define

$$\int_\beta^\alpha f := -\int_\alpha^\beta f \quad \text{and} \quad \int_\alpha^\alpha f := 0$$

Theorem 3.2.10. *If $f \in \mathcal{R}[a,b]$ and if α, β, γ are any numbers in $[a,b]$, then*

$$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f \tag{3.12}$$

in the sense that the existence of any two of these integrals implies the existence of the third integral and the equality (3.12).

Proof. If any two of the numbers α, β, γ are equal, then (3.12) holds. Thus we may suppose that all three of these numbers are distinct.

For the sake of symmetry, we introduce the expression

$$L(\alpha,\beta,\gamma) := \int_\alpha^\beta f + \int_\beta^\gamma f + \int_\gamma^\alpha f$$

It is clear that (3.12) holds if and only if $L(\alpha,\beta,\gamma) = 0$. Therefore, to establish the assertion, we need to show that $L = 0$ for all six permutations of the arguments α, β, and γ.

We note that the Additivity Theorem 3.2.9 implies that $L(\alpha,\beta,\gamma) = 0$ when $\alpha < \gamma < \beta$. But it is easily seen that both $L(\beta,\gamma,\alpha)$ and $L(\gamma,\alpha,\beta)$ are equal to $L(\alpha,\beta,\gamma)$. Moreover, the numbers

$$L(\beta,\alpha,\gamma), \quad L(\alpha,\gamma,\beta), \quad \text{and} \quad L(\gamma,\beta,\alpha)$$

are all equal to $-L(\alpha,\beta,\gamma)$. Therefore, L vanishes for all possible configurations of these three points. $\square$

EXERCISES

1. Let $f : [a,b] \to \mathbb{R}$. Show that $f \notin \mathcal{R}[a,b]$ if and only if there exists $\varepsilon_0 > 0$ such that for every $n \in \mathbb{N}$ there exist tagged partitions $\dot{\mathcal{P}}_n$ and $\dot{\mathcal{Q}}_n$ with $||\dot{\mathcal{P}}_n|| < 1/n$ and $||\dot{\mathcal{Q}}_n|| < 1/n$ such that $\left|S\left(f;\dot{\mathcal{P}}_n\right) - S\left(f;\dot{\mathcal{Q}}_n\right)\right| \geq \varepsilon_0$.

2. Consider the function h defined by $h(x) := x+1$ for $x \in [0,1]$ is rational, and $h(x) := 0$ for $x \in [0,1]$ is irrational. Show that h is not Riemann integrable.

3. Let $H(x) := k$ for $x = 1/k (k \in \mathbb{N})$ and $H(x) := 0$ elsewhere on

$[0, 1]$. Use Exercise 1 , or the argument in 3.2.2 (b), to show that H is not Riemann integrable.

4. If $\alpha(x) := -x$ and $\omega(x) := x$ and if $\alpha(x) \le f(x) \le \omega(x)$ for all $x \in [0, 1]$, does it follow from the Squeeze Theorem 3.2.3 that $f \in \mathcal{R}[0, 1]$?

5. If J is any subinterval of $[a, b]$ and if $\varphi_J(x) := 1$ for $x \in J$ and $\varphi_J(x) := 0$ elsewhere on $[a, b]$, we say that φ_J is an elementary step function on $[a, b]$. Show that every step function is a linear combination of elementary step functions.

6. If $\psi : [a, b] \to \mathbb{R}$ takes on only a finite number of distinct values, is ψ a step function?

7. If $S(f; \dot{\mathcal{P}})$ is any Riemann sum of $f : [a, b] \to \mathbb{R}$, show that there exists a step function $\varphi : [a, b] \to \mathbb{R}$ such that $\int_a^b \varphi = S(f; \dot{\mathcal{P}})$.

8. Suppose that f is continuous on $[a, b]$, that $f(x) \ge 0$ for all $x \in [a, b]$ and that $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

9. Show that the continuity hypothesis in the preceding exercise cannot be dropped.

10. If f and g are continuous on $[a, b]$ and if $\int_a^b f = \int_a^b g$, prove that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

11. If f is bounded by M on $[a, b]$ and if the restriction of f to every interval $[c, b]$ where $c \in (a, b)$ is Riemann integrable, show that $f \in \mathcal{R}[a, b]$ and that $\int_c^b f \to \int_a^b f$ as $c \to a+$. [Hint: Let $\alpha_c(x) := -M$ and $\omega_c(x) := M$ for $x \in [a, c)$ and $\alpha_c(x) := \omega_c(x) := f(x)$ for $x \in [c, b]$. Apply the Squeeze Theorem 3.2.3 for c sufficiently near a.]

12. Show that $g(x) := \sin(1/x)$ for $x \in (0,1]$ and $g(0) := 0$ belongs to $\mathcal{R}[0,1]$.

13. Give an example of a function $f : [a,b] \to \mathbb{R}$ that is in $\mathcal{R}[c,b]$ for every $c \in (a,b)$ but which is not in $\mathcal{R}[a,b]$.

14. Suppose that $f : [a,b] \to \mathbb{R}$, that $a = c_0 < c_1 < \cdots < c_m = b$ and that the restrictions of f to $[c_{i-1}, c_i]$ belong to $\mathcal{R}\left[c_{i-1}, c_i\right]$ for $i = 1, \ldots, m$. Prove that $f \in \mathcal{R}[a,b]$ and that the formula in Corollary 3.2.4 holds.

15. If f is bounded and there is a finite set E such that f is continuous at every point of $[a,b]\backslash E$, show that $f \in \mathcal{R}[a,b]$.

16. If f is continuous on $[a,b], a < b$, show that there exists $c \in [a,b]$ such that we have $\int_a^b f = f(c)(b-a)$. This result is sometimes called the Mean Value Theorem for Integrals.

17. If f and g are continuous on $[a,b]$ and $g(x) > 0$ for all $x \in [a,b]$, show that there exists $c \in [a,b]$ such that $\int_a^b fg = f(c) \int_a^b$ g. Show that this conclusion fails if we do not have $g(x) > 0$. (Note that this result is an extension of the preceding exercise.)

18. Let f be continuous on $[a,b]$, let $f(x) \geq 0$ for $x \in [a,b]$, and let $M_n := \left(\int_a^b f^n\right)^{1/n}$. Show that $\lim\left(M_n\right) = \sup\{f(x) : x \in [a,b]\}$.

19. Suppose that $a > 0$ and that $f \in \mathcal{R}[-a,a]$.

    (a) If f is even (that is, if $f(-x) = f(x)$ for all $x \in [0,a]$, show that $\int_{-a}^a f = 2\int_0^a f$.

    (b) If f is odd (that is, if $f(-x) = -f(x)$ for all $x \in [0,a]$), show that $\int_{-a}^a f = 0$.

20. If f is continuous on $[-a,a]$, show that $\int_{-a}^a f\left(x^2\right) dx = 2\int_0^a f\left(x^2\right) dx$.

3.3 The Fundamental Theorem

We will now explore the connection between the notions of the derivative and the integral. In fact, there are two theorems relating to this problem: one has to do with integrating a derivative, and the other with differentiating an integral. These theorems, taken together, are called the Fundamental Theorem of Calculus. Roughly stated, they imply that the operations of differentiation and integration are inverse to each other. However, there are some subtleties that should not be overlooked.

The Fundamental Theorem (First Form)

The First Form of the Fundamental Theorem provides a theoretical basis for the method of calculating an integral that the reader learned in calculus. It asserts that if a function f is the derivative of a function F, and if f belongs to $\mathcal{R}[a,b]$, then the integral $\int_a^b f$ can be calculated by means of the evaluation $F|_a^b := F(b) - F(a)$. A function F such that $F'(x) = f(x)$ for all $x \in [a,b]$ is called an **antiderivative** or a **primitive of** f on $[a,b]$. Thus, when f has an antiderivative, it is a very simple matter to calculate its integral.

In practice, it is convenient to allow some exceptional points c where $F'(c)$ does not exist in $\mathbb{R}$, or where it does not equal $f(c)$. It turns out that we can permit a finite number of such exceptional points.

Theorem 3.3.1. Fundamental Theorem of Calculus (First Form) *Suppose there is a finite set E in $[a,b]$ and functions $f, F := [a,b] \to \mathbb{R}$ such that:*

(a) F is continuous on $[a,b]$,

(b) $F'(x) = f(x)$ *for all* $x \in [a,b]\backslash E$,

(c) f *belongs to* $\mathcal{R}[a,b]$.

Then we have

$$\int_a^b f = F(b) - F(a). \tag{3.13}$$

Proof. We will prove the theorem in the case where $E := \{a, b\}$. The general case can be obtained by breaking the interval into the union of a finite number of intervals.

Let $\varepsilon > 0$ be given. Since $f \in \mathcal{R}[a,b]$ by assumption (c), there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, then

$$\left| S(f;\dot{\mathcal{P}}) - \int_a^b f \right| < \varepsilon. \tag{3.14}$$

If the subintervals in $\dot{\mathcal{P}}$ are $[x_{i-1}, x_i]$, then the Mean Value Theorem 1.5.5 applied to F on $[x_{i-1}, x_i]$ implies that there exists $u_i \in (x_{i-1}, x_i)$ such that

$$F(x_i) - F(x_{i-1}) = F'(u_i) \cdot (x_i - x_{i-1}) \quad \text{for} \quad i = 1, \ldots, n.$$

If we add these terms, note the telescoping of the sum, and use the fact that
$F'(u_i) = f(u_i)$, we obtain

$$F(b)-F(a)=\sum_{i=1}^{n}\left(F\left(x_i\right)-F\left(x_{i-1}\right)\right)=\sum_{i=1}^{n} f\left(u_i\right)\left(x_i-x_{i-1}\right)$$

Now let $\dot{\mathcal{P}}_u:=\left\{\left(\left[x_{i-1}, x_i\right], u_i\right)\right\}_{i=1}^{n}$, so the sum on the right equals $S\left(f ; \dot{\mathcal{P}}_u\right)$. If we substitute $F(b)-F(a)=S\left(f ; \dot{\mathcal{P}}_u\right)$ into (3.14), we conclude that

$$\left|F(b)-F(a)-\int_a^b f\right|<\varepsilon$$

But, since $\varepsilon>0$ is arbitrary, we infer that equation (3.13) holds.

$\square$

Remark 3.3.1. If the function F is differentiable at every point of $[a, b]$, then (by Theorem 1.5.1) hypothesis (a) is automatically satisfied. If f is not defined for some point $c \in E$, we take $f(c):=0$. Even if F is differentiable at every point of $[a, b]$, condition (c) is not automatically satisfied, since there exist functions F such that F' is not Riemann integrable.

Example 3.3.2.

(a) If $F(x):=\frac{1}{2} x^2$ for all $x \in[a, b]$, then $F'(x)=x$ for all $x \in[a, b]$. Further, $f=F'$ is continuous so it is in $\mathcal{R}[a, b]$. Therefore the Fundamental Theorem (with $E=\emptyset$) implies that

$$\int_a^b x d x=F(b)-F(a)=\frac{1}{2}\left(b^2-a^2\right) .$$

(b) If $G(x):=\operatorname{Arctan} x$ for $x \in[a, b]$, then $G'(x)=1/\left(x^2+1\right)$ for all

$x \in [a,b]$; also G' is continuous, so it is in $\mathcal{R}[a,b]$. Therefore the Fundamental Theorem (with $E = \emptyset$) implies that

$$\int_a^b \frac{1}{x^2+1} dx = \operatorname{Arctan} b - \operatorname{Arctan} a$$

(c) If $A(x) := |x|$ for $x \in [-10, 10]$, then $A'(x) = -1$ if $x \in [-10, 0)$, $A'(x) = +1$ for $x \in (0, 10]$ and $A(x)$ is not differentiable at $x = 0$. Signum function *sgn* is defined as follows,

$$sgn(x) := \begin{cases} +1 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -1 & \text{for } x > 0, \end{cases}$$

So we have $A'(x) = \operatorname{sgn}(x)$ for all $x \in [-10, 10] \backslash \{0\}$. Since the signum function is a step function, it belongs to $\mathcal{R}[-10, 10]$. Therefore the Fundamental Theorem (with $E = \{0\}$, finite set) implies that

$$\int_{-10}^{10} \operatorname{sgn}(x) dx = A(10) - A(-10) = 10 - 10 = 0.$$

(d) If $H(x) := 2\sqrt{x}$ for $x \in [0, b]$, then H is continuous on $[0, b]$ and $H'(x) = 1/\sqrt{x}$ for $x \in (0, b]$. Since $h := H'$ is not bounded on $(0, b]$, it does not belong to $\mathcal{R}[0, b]$ no matter how we define $h(0)$. Therefore, the Fundamental Theorem 3.3.1 does not apply.

(e) Let $K(x) := x^2 \cos\left(1/x^2\right)$ for $x \in (0, 1]$ and let $K(0) := 0$. It follows from the Product Rule 1.5.2(c) and the Chain Rule 1.5.4 that

$$K'(x) = 2x \cos\left(1/x^2\right) + (2/x) \sin\left(1/x^2\right) \quad \text{for} \quad x \in (0, 1].$$

Further it can be shown that $K'(0) = 0$. Thus K is continuous and differentiable at every point of $[0, 1]$. Since it can be seen that the function K' is not bounded on $[0, 1]$, it does not belong to $\mathcal{R}[0, 1]$ and the Fundamental Theorem 3.3.1 does not apply to K'.

The Fundamental Theorem (Second Form)

We now turn to the Fundamental Theorem (Second Form) in which we wish to differentiate an integral involving a variable upper limit.

Definition 27. If $f \in \mathcal{R}[a, b]$, then the function defined by

$$F(z) := \int_a^z f \quad \text{for} \quad z \in [a, b], \tag{3.15}$$

is called the **indefinite integral** of f with **basepoint** a. (Sometimes a point other than a is used as a basepoint.)

We will first show that if $f \in \mathcal{R}[a, b]$, then its indefinite integral F satisfies a Lipschitz condition; hence F is continuous on $[a, b]$.

Theorem 3.3.3. *The indefinite integral F defined by $F(z) := \int_a^z f$ for $z \in [a, b]$, is continuous on $[a, b]$. In fact, if $|f(x)| \leq M$ for all $x \in [a, b]$, then $|F(z) - F(w)| \leq M|z - w|$ for all $z, w \in [a, b]$.*

Proof. The Additivity Theorem 3.2.9 implies that if $z, w \in [a, b]$ and $w \leq z$, then

$$F(z) = \int_a^z f = \int_a^w f + \int_w^z f = F(w) + \int_w^z f$$

whence we have

$$F(z) - F(w) = \int_w^z f.$$

Now if $-M \leq f(x) \leq M$ for all $x \in [a, b]$, then properties of integrals implies that

$$-M(z-w) \leq \int_w^z f \leq M(z-w),$$

whence it follows that

$$|F(z) - F(w)| \leq \left| \int_w^z f \right| \leq M|z-w|,$$

as asserted.

□

We will now show that the indefinite integral F is differentiable at any point where f is continuous.

Theorem 3.3.4. Fundamental Theorem of Calculus (**Second Form**) *Let $f \in \mathcal{R}[a, b]$ and let f be continuous at a point $c \in [a, b]$. Then the indefinite integral, defined by (3.15), is differentiable at c and $F'(c) = f(c)$.*

Proof. We will suppose that $c \in [a, b)$ and consider the right-hand derivative of F at c. Since f is continuous at c, given $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$

such that if $c \leq x < c + \eta_\varepsilon$, then

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon. \tag{3.16}$$

Let h satisfy $0 < h < \eta_e$. The Additivity Theorem 3.2.9 implies that f is integrable on the intervals $[a, c], [a, c + h]$ and $[c, c + h]$ and that

$$F(c + h) - F(c) = \int_c^{c+h} f$$

Now on the interval $[c, c+h]$ the function f satisfies inequality (3.16), so that we have

$$(f(c) - \varepsilon) \cdot h \leq F(c + h) - F(c) = \int_c^{c+h} f \leq (f(c) + \varepsilon) \cdot h$$

If we divide by $h > 0$ and subtract $f(c)$, we obtain

$$\left| \frac{F(c + h) - F(c)}{h} - f(c) \right| \leq \varepsilon.$$

But, since $\varepsilon > 0$ is arbitrary, we conclude that the right-hand limit is given by

$$\lim_{x \to 0+} \frac{F(c + h) - F(c)}{h} = f(c)$$

It is proved in the same way that the left-hand limit of this difference quotient also equals $f(c)$ when $c \in (a, b]$, whence the assertion follows. $\square$

Theorem 3.3.5. *If f is continuous on $[a,b]$, then the indefinite integral F, defined by (3.15), is differentiable on $[a,b]$ and $F'(x) = f(x)$ for all $x \in [a,b]$.*

Remark 3.3.2. If f is continuous on $[a,b]$, then its indefinite integral is an antiderivative of f. We will now see that, in general, the indefinite integral need not be an antiderivative (either because the derivative of the indefinite integral does not exist or does not equal $f(x)$).

Example 3.3.6.

(a) If $f(x) := \operatorname{sgn} x$ on $[-1,1]$, then $f \in \mathcal{R}[-1,1]$ and has the indefinite integral

$F(z) := \int_{-1}^{z} f = \int_{-1}^{z} \operatorname{sgn} x dx$

If $z \leq 0$ then $F(z) := \int_{-1}^{z}(-1)dx = [-x]_{-1}^{z} = -z - 1$

If $z > 0$ then

$$\begin{aligned} F(z) :&= \int_{-1}^{0}(-1)dx + \int_{0}^{z}(1)dx \\ &= [-x]_{-1}^{0} + [x]_{0}^{z} = -1 + z \text{ (We done this using Theorem 3.1.5)} \end{aligned}$$

$F(x) := |x| - 1$ with the basepoint -1. However, since $F'(0)$ does not exist, F is not an antiderivative of f on $[-1,1]$.

(b) If h denotes Thomae's function then its indefinite integral $H(x) := \int_{0}^{x} h$ is identically 0 on $[0,1]$. Here, the derivative of this indefinite integral exists at every point and $H'(x) = 0$. But $H'(x) \neq h(x)$

whenever $x \in \mathbb{Q} \cap [0,1]$, so that H is not an antiderivative of h on $[0,1]$.

Substitution Theorem

The next theorem provides the justification for the "change of variable" method that is often used to evaluate integrals.

Theorem 3.3.7. Substitution Theorem *Let $J := [\alpha, \beta]$ and let $\varphi : J \to \mathbb{R}$ have a continuous derivative on J. If $f : I \to \mathbb{R}$ is continuous on an interval I containing $\varphi(J)$, then*

$$\int_{\alpha}^{\beta} f(\varphi(t)) \cdot \varphi'(t)dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx.$$

Proof. Let f and ϕ be two functions satisfying above hypothesis that f is continuous on I and $\phi^{'}$ is integrable on $[\alpha, \beta]$. Then the function $f(\phi(t))\phi^{'}(t)$ is also integrable on $[\alpha, \beta]$. Hence the integrals $\int_{\alpha}^{\beta} f(\phi(t))\phi^{'}(t)$ and $\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx$ in fact exist. Since f is continuous on I, it has an antiderivative F by Theorem 3.3.5, where $F(x) = \int_{\phi(\alpha)}^{x} f(t)dt$. The composite function $F \circ \phi$ is then defined. Since ϕ is differentiable, combining the chain rule, Theorem 1.5.4 we get,

$$(F \circ \phi)^{'}(t) = F^{'}(\phi(t))\phi^{'}(t) = f(\phi(t))\phi^{'}(t).$$

Applying Fundamental Theorem 3.3.1 for $(F \circ \phi)^{'}(t) = f(\phi(t))\phi^{'}(t)$,

$$\int_{\alpha}^{\beta} f(\phi(t))\phi^{'}(t)dt = \int_{\alpha}^{\beta} (F \circ \phi)^{'}(t)dt$$

$$
\begin{aligned}
&= (F \circ \phi)(\beta) - (F \circ \phi)(\alpha) \\
&= F(\phi(\beta)) - F(\phi(\alpha)) \\
&= \int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx
\end{aligned}
$$

$\square$

Example 3.3.8.

(a) Consider the integral $\int_1^4 \frac{\sin\sqrt{t}}{\sqrt{t}}dt$.

Here we substitute $\varphi(t) := \sqrt{t}$ for $t \in [1,4]$ so that $\varphi'(t) = 1/(2\sqrt{t})$ is continuous on $[1,4]$. If we let $f(x) := 2\sin x$, then the integrand has the form $(f \circ \varphi) \cdot \varphi'$ and the Substitution Theorem 3.3.7 implies that the integral equals $\int_1^2 2\sin x dx = -\ 2\cos x]_1^2 = 2(\cos 1 - \cos 2)$.

(b) Consider the integral $\int_0^4 \frac{\sin\sqrt{t}}{\sqrt{t}}dt$.

Since $\varphi(t) := \sqrt{t}$ does not have a continuous derivative on $[0,4]$, the Substitution Theorem 3.3.7 is not applicable, at least with this substitution. (In fact, it is not obvious that this integral exists; however, we can apply Exercise 2.2.11 to obtain this conclusion. We could then apply the Fundamental Theorem 3.3.1 to $F(t) := -2\cos\sqrt{t}$ with $E := \{0\}$ to evaluate this integral.)

Lebesgue's Integrability Criterion

We will now present a statement of the definitive theorem due to Henri Lebesgue (1875-1941) giving a necessary and sufficient condition for a function to be Riemann integrable, and will give some applications of this theorem. In order to state this result, we need to introduce the important notion of a null set.

Warning Some people use the term "null set" as a synonym for the terms "empty set" or "void set" referring to $\emptyset$ (= the set that has no elements). However, we will always use the term "null set" in conformity with our next definition, as is customary in the theory of integration.

Definition 28.

(a) A set $Z \subset \mathbb{R}$ is said to be a **null set** if for every $\varepsilon > 0$ there exists a countable collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ of open intervals such that

$$Z \subseteq \bigcup_{k=1}^{\infty}(a_k, b_k) \quad \text{and} \quad \sum_{k=1}^{\infty}(b_k - a_k) \leq \varepsilon.$$

(b) If $Q(x)$ is a statement about the point $x \in I$, we say that $Q(x)$ holds **almost everywhere** on I (or for almost every $x \in I$), if there exists a null set $Z \subset I$ such that $Q(x)$ holds for all $x \in I \backslash Z$. In this case we may write

$$Q(x) \quad \text{for a.e.} \quad x \in I.$$

Note 3.3.9.

1. It is trivial that any subset of a null set is also a null set.

2. The union of two null sets is a null set.

Example 3.3.10. The $\mathbb{Q}_1$ of rational numbers in $[0, 1]$ is a null set. We enumerate
$\mathbb{Q}_1 = \{r_1, r_2, \ldots\}$. Given $\varepsilon > 0$, note that the open interval $J_1 := (r_1 - \varepsilon/4, r_1 + \varepsilon/4)$ contains r_1 and has length $\varepsilon/2$; also the open interval $J_2 := (r_2 - \varepsilon/8, r_2 + \varepsilon/8)$ contains r_2 and has length $\varepsilon/4$. In general, the open interval

$$J_k := \left(r_k - \frac{\varepsilon}{2^{k+1}}, r_k + \frac{\varepsilon}{2^{k+1}}\right)$$

contains the point r_k and has length $\varepsilon/2^k$. Therefore, the union $\bigcup_{k=1}^{\infty} J_k$ of these open intervals contains every point of $\mathbb{Q}_1$; moreover, the sum of the lengths is $\sum_{k=1}^{\infty}\left(\varepsilon/2^k\right) = \varepsilon$. Since $\varepsilon > 0$ is arbitrary. $\mathbb{Q}_1$ is a null set.

Note 3.3.11.

- Every countable set is a null set.
- there exist uncountable null sets in $\mathbb{R}$;(for example, the Cantor set)

We now state Lebesgue's Integrability Criterion. It asserts that a bounded function on an interval is Riemann integrable if and only if its points of discontinuity form a null set.

Theorem 3.3.12. Lebesgue$'$s Integrability Criterion *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous almost everywhere on $[a, b]$.*

We will apply Lebesgue's Theorem here to some specific functions, and show that some of our previous results follow immediately from it. We shall also use this theorem to obtain the important Composition and Product Theorems.

Example 3.3.13.

(a) Define a step function $g : [0, 3] \to \mathbb{R}$ such that

$$g(x) := \begin{cases} 2 & \text{for } 0 \leq x \leq 1, \\ 3 & \text{for } 1 < x \leq 3, \end{cases}$$

g is continuous at every point except the point $x = 1$. Therefore it follows from the Lebesgue Integrability Criterion that g is Riemann integrable.

(b) Since the set of points of discontinuity of a monotone function is countable, we conclude that: Every monotone function on $[a, b]$ is Riemann integrable.

(c) Define a function $G : [0, 1] \to \mathbb{R}$ as

$$G(x) := \begin{cases} 1/\text{n} & \text{for } x = 1/n,\ n \in \mathbb{N} \\ 0 & \text{else where on } [0, 1] \end{cases}$$

G is discontinuous precisely at the points $D := \{1, 1/2, \ldots, 1/n, \ldots\}$. Since this is a countable set, it is a null set and Lebesgue's Criterion implies that G is Riemann integrable.

(d) The Dirichlet function not to be Riemann integrable.

Note that it is discontinuous at every point of $[0, 1]$. Since the interval $[0, 1]$ is not a null set, Lebesgue's Criterion yields the same conclusion.

(e) Let $h : [0, 1] \to \mathbb{R}$ be Thomae's function. Then h is continuous at every irrational number and is discontinuous at every rational number in $[0, 1]$. It is discontinuous on a null set, so Lebesgue's Criterion implies that Thomae's function is Riemann integrable on $[0, 1]$.

We now obtain a result that will enable us to take other combinations of Riemann integrable functions.

Theorem 3.3.14. Composition Theorem *Let $f \in \mathcal{R}[a,b]$ with $f([a,b]) \subseteq [c,d]$ and let $\varphi : [c,d] \to \mathbb{R}$ be continuous. Then the composition $\varphi \circ f$ belongs to $\mathcal{R}[a,b]$.*

Proof. If f is continuous at a point $u \in [a,b]$, then $\varphi \circ f$ is also continuous at u. Since the set D of points of discontinuity of f is a null set, it follows that the set $D_1 \subseteq D$ of points of discontinuity of $\varphi \circ f$ is also a null set. Therefore the composition $\varphi \circ f$ also belongs to $\mathcal{R}[a,b]$. $\square$

It will be seen in Exercise 22 that the hypothesis that φ is continuous cannot be dropped. The next result is a corollary of the Composition Theorem.

Corollary 3.3.3. *Suppose that $f \in \mathcal{R}[a,b]$. Then its absolute value $|f|$ is in $\mathcal{R}[a,b]$, and*

$$\left| \int_a^b f \right| \leq \int_a^b |f| \leq M(b-a)$$

where $|f(x)| \leq M$ for all $x \in [a,b]$.

Proof. From Boundedness Theorem, we have seen that if f is integrable, then there exists M such that $|f(x)| \leq M$ for all $x \in [a,b]$. Let $\varphi(t) := |t|$ for $t \in [-M,M]$; then the Composition Theorem implies that $|f| = \varphi \circ f \in \mathcal{R}[a,b]$.
We have $-|f| \leq f \leq |f|$.

So $\int_a^b -|f| \le \int_a^b f \le \int_a^b |f|$ which implies that $\left|\int_a^b f\right| \le \int_a^b |f|$.

Also we have $|f(x)| \le M$. Hence $\int_a^b |f| \le \int_a^b M = M(b-a)$. □

Theorem 3.3.15. The Product Theorem *If f and g belong to $\mathcal{R}[a,b]$, then the product fg belongs to $\mathcal{R}[a,b]$.*

Proof. If $\varphi(t) := t^2$ for $t \in [-M, M]$, it follows from the Composition Theorem that $f^2 = \varphi \circ f$ belongs to $\mathcal{R}[a,b]$. Similarly, $(f+g)^2$ and g^2 belong to $\mathcal{R}[a,b]$. But since we can write the product as

$$fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right],$$

it follows that $fg \in \mathcal{R}[a,b]$. □

Integration by Parts

We will conclude this section with a rather general form of Integration by Parts for the Riemann integral, and Taylor's Theorem with the Remainder.

Theorem 3.3.16. Integration by Parts *Let F, G be differentiable on $[a,b]$ and let $f := F'$ and $g := G'$ belong to $\mathcal{R}[a,b]$. Then*

$$\int_a^b fG = FG\Big|_a^b - \int_a^b Fg$$

Proof. By Theorem 1.5.2 (c), the derivative $(FG)'$ exists on $[a,b]$ and

$$(FG)' = F'G + FG' = fG + Fg.$$

Since F, G are continuous and f, g belong to $\mathcal{R}[a, b]$, the Product Theorem 3.3.15 implies that fG and Fg are integrable. Therefore the Fundamental Theorem 3.3.1 implies that

$$FG|_a^b = \int_a^b (FG)' = \int_a^b fG + Fg = \int_a^b fG + \int_a^b Fg.$$

$$\Rightarrow \int_a^b fG = FG|_a^b - \int_a^b Fg.$$

$\square$

We close this section with a version of Taylor's Theorem for the Riemann Integral.

Theorem 3.3.17. *Taylor's Theorem with the Remainder Suppose that $f', \ldots, f^{(n)}, f^{(n+1)}$ exist on $[a, b]$ and that $f^{(n+1)} \in \mathcal{R}[a, b]$. Then we have*

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + R_n, \tag{3.17}$$

where the remainder is given by

$$R_n = \frac{1}{n!} \int_a^b f^{(n+1)}(t) \cdot (b-t)^n dt. \tag{3.18}$$

Proof. Apply Integration by Parts to equation (3.18), with $F(t) := f^{(n)}(t)$ and $G(t) := (b-t)^n/n!$, so that $g(t) = -(b-t)^{n-1}/(n-1)!$, to get

$$\begin{aligned} R_n &= \frac{1}{n!} f^{(n)}(t) \cdot (b-t)^n \Big|_{t=a}^{t=b} + \frac{1}{(n-1)!} \int_a^b f^{(n)}(t) \cdot (b-a)^{n-1} dt \\ &= -\frac{f^{(n)}(a)}{n!} \cdot (b-a)^n + \frac{1}{(n-1)!} \int_a^b f^{(n)}(t) \cdot (b-t)^{n-1} dt \end{aligned}$$

If we continue to integrate by parts in this way, we obtain (3.17). □

EXERCISES

1. Extend the proof of the Fundamental Theorem 3.3.1 to the case of an arbitrary finite set E.

2. If $n \in \mathbb{N}$ and $H_n(x) := x^{n+1}/(n+1)$ for $x \in [a,b]$, show that the Fundamental Theorem 3.3.1 implies that $\int_a^b x^n dx = \left(b^{n+1} - a^{n+1}\right)/(n+1)$. What is the set E here?

3. If $g(x) := x$ for $|x| \geq 1$ and $g(x) := -x$ for $|x| < 1$ and if $G(x) := \frac{1}{2}\left|x^2 - 1\right|$, show that $\int_{-2}^3 g(x)dx = G(3) - G(-2) = 5/2$.

4. Let $B(x) := -\frac{1}{2}x^2$ for $x < 0$ and $B(x) := \frac{1}{2}x^2$ for $x \geq 0$. Show that $\int_a^b |x|dx = B(b) - B(a)$.

5. Let $f : [a,b] \to \mathbb{R}$ and let $C \in \mathbb{R}$.

   (a) If $\Phi : [a,b] \to \mathbb{R}$ is an antiderivative of f on $[a,b]$, show that $\Phi_C(x) := \Phi(x) + C$ is also

   an antiderivative of f on $[a,b]$.

(b) If Φ_1 and Φ_2 are antiderivatives of f on $[a, b]$, show that $\Phi_1 - \Phi_2$ is a constant function on

$[a, b]$

6. If $f \in \mathcal{R}[a, b]$ and if $c \in [a, b]$, the function defined by $F_c(z) := \int_c^z f$ for $z \in [a, b]$ is called the indefinite integral of f with basepoint c. Find a relation between F_a and F_c.

7. We have seen in Example 3.1.9 that Thomae's function is in $\mathcal{R}[0, 1]$ with integral equal to 0 . Can the Fundamental Theorem 3.3.1 be used to obtain this conclusion? Explain your answer.

8. Let $F(x)$ be defined for $x \geq 0$ by $F(x) := (n-1)x - (n-1)n/2$ for $x \in [n-1, n), n \in \mathbb{N}$. Show that F is continuous and evaluate $F'(x)$ at points where this derivative exists. Use this result to evaluate $\int_a^b \lfloor x \rfloor dx$ for $0 \leq a < b$, where $\lfloor x \rfloor$ denotes the greatest integer in x.

9. Let $f \in \mathcal{R}[a, b]$ and define $F(x) := \int_a^x f$ for $x \in [a, b]$.

   (a) Evaluate $G(x) := \int_c^x f$ in terms of F, where $c \in [a, b]$.

   (b) Evaluate $H(x) := \int_x^b f$ in terms of F.

   (c) Evaluate $S(x) := \int_x^{\sin x} f$ in terms of F.

10. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and let $v : [c, d] \to \mathbb{R}$ be differentiable on $[c, d]$ with $v([c, d]) \subseteq [a, b]$. If we define $G(x) := \int_a^{v(x)} f$, show that $G'(x) = f(v(x)) \cdot v'(x)$ for all $x \in [c, d]$.

11. Find $F'(x)$ when F is defined on $[0, 1]$ by:

    (a) $F(x) := \int_0^{x^2} \left(1 + t^3\right)^{-1} dt$.

    (b) $F(x) := \int_{x^2}^{x} \sqrt{1 + t^2} dt$.

12. Let $f : [0, 3] \to \mathbb{R}$ be defined by $f(x) := x$ for $0 \le x < 1, f(x) := 1$ for $1 \le x < 2$ and $f(x) := x$ for $2 \le x \le 3$. Obtain formulas for $F(x) := \int_0^x f$ and sketch the graphs of f and F. Where is F differentiable? Evaluate $F'(x)$ at all such points.

13. The function g is defined on $[0, 3]$ by $g(x) := -1$ if $0 \le x < 2$ and $g(x) := 1$ if $2 \le x \le 3$. Find the indefinite integral $G(x) = \int_0^x g$ for $0 \le x \le 3$, and sketch the graphs of g and G. Does $G'(x) = g(x)$ for all x in $[0, 3]$?

14. Show there does not exist a continuously differentiable function f on $[0, 2]$ such that $f(0) = -1, f(2) = 4$, and $f'(x) \le 2$ for $0 \le x \le 2$. (Apply the Fundamental Theorem.)

15. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $c > 0$, define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := \int_{x-c}^{x+c} f(t)dt$. Show that g is differentiable on $\mathbb{R}$ and find $g'(x)$.

16. If $f : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$, show that $f(x) = 0$ for all $x \in [0, 1]$.

17. Use the following argument to prove the Substitution Theorem 3.3.7. Define $F(u) := \int_{\varphi(\alpha)}^u f(x)dx$ for $u \in I$, and $H(t) := F(\varphi(t))$ for $t \in J$. Show that $H'(t) = f(\varphi(t))\varphi'(t)$ for $t \in J$ and that

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx = F(\varphi(\beta)) = H(\beta) = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt.$$

18. Use the Substitution Theorem 3.3.7 to evaluate the following integrals.

    (a) $\int_0^1 t\sqrt{1 + t^2}dt$,

    (b) $\int_0^2 t^2 \left(1 + t^3\right)^{-1/2} dt = 4/3$,

(c) $\int_1^4 \frac{\sqrt{1+\sqrt{t}}}{\sqrt{t}} dt$,

(d) $\int_1^4 \frac{\cos\sqrt{t}}{\sqrt{t}} dt = 2(\sin 2 - \sin 1)$.

19. Explain why Theorem 3.3.7 cannot be applied to evaluate the following integrals, using the indicated substitution.

(a) $\int_0^4 \frac{\sqrt{t}dt}{1+\sqrt{t}}$ $\varphi(t) = \sqrt{t}$,

(b) $\int_0^4 \frac{\cos\sqrt{t}dt}{\sqrt{t}}$ $\varphi(t) = \sqrt{t}$,

(c) $\int_{-1}^1 \sqrt{1+2|t|}dt$ $\varphi(t) = |t|$,

(d) $\int_0^1 \frac{dt}{\sqrt{1-t^2}}$ $\varphi(t) = \text{Arcsin}\, t$.

20. (a) If Z_1 and Z_2 are null sets, show that $Z_1 \cup Z_2$ is a null set.

(b) More generally, if Z_n is a null set for each $n \in \mathbb{N}$, show that $\bigcup_{n=1}^{\infty} Z_n$ is a null set.[Hint: Given $\varepsilon > 0$ and $n \in \mathbb{N}$, let $\{J_k^n : k \in \mathbb{N}\}$ be a countable collection of open intervals whose union contains Z_n and the sum of whose lengths is $\leq \varepsilon/2^n$. Now consider the countable collection $\{J_k^n : k \in \mathbb{N}\}$.]

21. Let $f, g \in \mathcal{R}[a, b]$.

(a) If $t \in \mathbb{R}$, show that $\int_a^b (tf \pm g)^2 \geq 0$.

(b) Use (a) to show that $2\left|\int_a^b fg\right| \leq t\int_a^b f^2 + (1/t)\int_a^b g^2$ for $t > 0$.

(c) If $\int_a^b f^2 = 0$, show that $\int_a^b fg = 0$.

(d) Now prove that $\left|\int_a^b fg\right|^2 \leq \left(\int_a^b |fg|\right)^2 \leq \left(\int_a^b f^2\right) \cdot \left(\int_a^b g^2\right)$. This inequality is called Cauchy-Bunyakovsky-Schwarz Inequality (or simply the Schwarz Inequality).

22. Let $h : [0, 1] \to \mathbb{R}$ be Thomae's function and let sgn be the signum function. Show that the composite function $sgn \circ h$ is not Riemann integrable on $[0, 1]$.

Chapter 4

POINTWISE AND UNIFORM CONVERGENCE, INTERCHANGE OF LIMITS AND SERIES OF FUNCTIONS

In Section 4.1 we will introduce two different notions of convergence for a sequence of functions: pointwise convergence and uniform convergence. The latter type of convergence is very important, and will be the main focus of our attention. The reason for this focus is the fact that, as is shown in Section 4.2, uniform convergence "preserves" certain properties in the sense that if each term of a uniformly convergent sequence of functions possesses these properties, then the limit function also possesses the properties.

4.1 Pointwise and Uniform Convergence

Let $A \subseteq \mathbb{R}$ be given and suppose that for each $n \in \mathbb{N}$ there is a function $f_n : A \to \mathbb{R}$; we shall say that (f_n) is a **sequence of functions** on A to $\mathbb{R}$. Clearly, for each $x \in A$, such a sequence gives rise to a sequence of real numbers, namely the sequence

$$(f_n(x))$$

obtained by evaluating each of the functions at the point x. For certain values of $x \in A$ the sequence $(f_n(x))$ may converge, and for other values of $x \in A$ this sequence may diverge. For each $x \in A$ for which the sequence $(f_n(x))$ converges, there is a uniquely determined real number $\lim (f_n(x))$. In general, the value of this limit, when it exists, will depend on the choice of the point $x \in A$. Thus, there arises in this way a function whose domain consists of all numbers $x \in A$ for which the sequence $(f_n(x))$ converges.

Definition 29. Let (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, let $A_0 \subseteq A$, and let $f : A_0 \to \mathbb{R}$. We say that the **sequence** (f_n) **converges on** A_0 **to** f if, for each $x \in A_0$, the sequence $(f_n(x))$ converges to $f(x)$ in $\mathbb{R}$. In this case we call f the **limit on** A_0 **of the sequence** (f_n). When such a function f exists, we say that the sequence (f_n) **is convergent on** A_0, or that (f_n) **converges pointwise on** A_0.

Since a sequence in $\mathbb{R}$ can have atmost one limit, except for a possible modification of the domain A_0, the limit function is uniquely determined. Ordinarily we choose A_0 to be the largest set possible; that is, we take A_0 to be the set of all $x \in A$ for which the sequence $(f_n(x))$ is convergent in $\mathbb{R}$.

In order to symbolize that the sequence (f_n) converges on A_0 to f, we sometimes write

$$f = \lim(f_n) \text{ on } A_0, \text{ or } f_n \to f \text{ on } A_0.$$

Sometimes, when f_n and f are given by formulas, we write

$$f(x) = \lim f_n(x) \text{ for } x \in A_0, \quad \text{ or } f_n(x) \to f(x) \text{ for } x \in A_0.$$

Example 4.1.1.

(a) $\lim(x/n) = 0$ for $x \in \mathbb{R}$.

For $n \in \mathbb{N}$, let $f_n(x) := x/n$ and let $f(x) := 0$ for $x \in \mathbb{R}$. We have $\lim(1/n) = 0$. Hence it follows from Theorem 1.2.2 that

$$\lim(f_n(x)) = \lim(x/n) = x\lim(1/n) = x \cdot 0 = 0$$

for all $x \in \mathbb{R}$. (See Figure 4.1.)

(b) $\lim(x^n)$.

Let $g_n(x) := x^n$ for $x \in \mathbb{R}, n \in \mathbb{N}$. (See Figure 4.1.) Clearly, if $x = 1$, then the sequence $(g_n(1)) = (1)$ converges to 1. We have $\lim(x^n) = 0$ for $0 \leq x < 1$ and it is readily seen that this is also true for $-1 < x < 0$. If $x = -1$, then $g_n(-1) = (-1)^n$, the sequence is divergent. Similarly, if $|x| > 1$, then the sequence (x^n) is not bounded, and so it is not convergent in $\mathbb{R}$. We conclude that if

$$g(x) := \begin{cases} 0 & \text{for} \quad -1 < x < 1 \\ 1 & \text{for} \quad x = 1 \end{cases}$$

then the sequence (g_n) converges to g on the set $(-1, 1]$.

(c) $\lim \left(\left(x^2 + nx \right) / n \right) = x$ for $x \in \mathbb{R}$.

Let $h_n(x) := \left(x^2 + nx \right) / n$ for $x \in \mathbb{R}, n \in \mathbb{N}$, and let $h(x) := x$ for $x \in \mathbb{R}$. (See Figure 4.2.) Since we have $h_n(x) = \left(x^2/n \right) + x$, it follows from Theorem 1.2.2 that $h_n(x) \to x = h(x)$ for all $x \in \mathbb{R}$.

(d) $\lim((1/n)\sin(nx + n)) = 0$ for $x \in \mathbb{R}$.

Let $F_n(x) := (1/n)\sin(nx + n)$ for $x \in \mathbb{R}, n \in \mathbb{N}$, and let $F(x) := 0$ for $x \in \mathbb{R}$. (See Figure 4.2.) Since $|\sin y| \leq 1$ for all $y \in \mathbb{R}$ we have

$$|F_n(x) - F(x)| = \left| \frac{1}{n} \sin(nx + n) \right| \leq \frac{1}{n}$$

for all $x \in \mathbb{R}$. Therefore it follows that $\lim (F_n(x)) = 0 = F(x)$ for all $x \in \mathbb{R}$. Note that, given any $\varepsilon > 0$, if n is sufficiently large, then $|F_n(x) - F(x)| < \varepsilon$ for all values of x simultaneously!

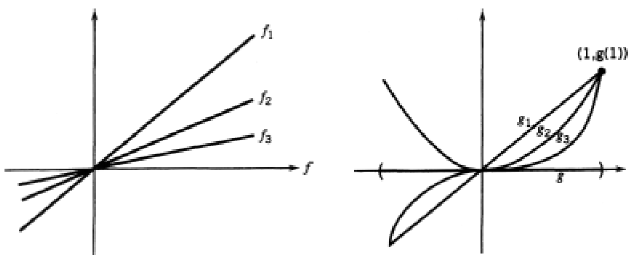

Figure 4.1: $f_n(x) = x/n$ and $g_n(x) = x^n$

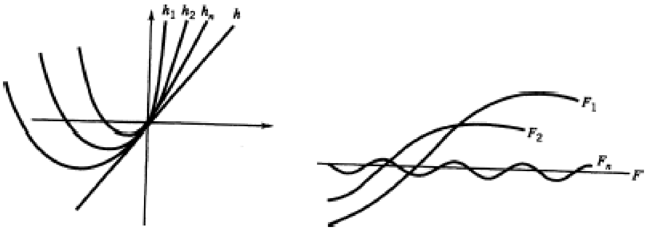

Figure 4.2: $h_n(x) = (x^2 + nx)/n$ and $F_n(x) = \sin(nx + n)/n$

Lemma 4.1.1. *A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ converges to a function*
$f : A_0 \to \mathbb{R}$ on A_0 if and only if for each $\varepsilon > 0$ and each $x \in A_0$ there is a natural number $K(\varepsilon, x)$ such that if $n \geq K(\varepsilon, x)$, then

$$|f_n(x) - f(x)| < \varepsilon.$$

Note 4.1.2.

1. Lemma 4.1.1 is equivalent to Definition 29.
   (Assume that for each $\varepsilon > 0$ and each $x \in A_0$ there is a natural number $K(\varepsilon, x)$ such that if $n \geq K(\varepsilon, x)$, then $|f_n(x) - f(x)| < \varepsilon$. So using definition of convergence of a sequence we can say that the sequence $(f_n(x)) \to f(x)$ for each $x \in A_0$.)

2. In part (c) of above Example $h_n(x) := \left(x^2 + nx\right)/n$ for $x \in \mathbb{R}$ and $h(x) := x$ for $x \in \mathbb{R}$. Then $|h_n(x) - h(x)| = \left|\dfrac{\left(x^2 + nx\right)}{n} - x\right| = \left|\dfrac{x^2}{n}\right|$. Now $|h_n(x) - h(x)| < \varepsilon \Rightarrow \dfrac{x^2}{n} < \varepsilon$

If $n > \dfrac{x^2}{\varepsilon}$ then $|h_n(x) - h(x)| < \varepsilon$. Hence choose $K(\varepsilon, x) =$ Natural number greater than $\dfrac{x^2}{n}$.

Uniform Convergence

Definition 30. A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ **converges uniformly on** $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ if for each $\varepsilon > 0$ there is a natural number $K(\varepsilon)$ (depending on ε but **not** on $x \in A_0$) such that if $n \geq K(\varepsilon)$, then

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all} \quad x \in A_0$$

In this case we say that the sequence (f_n) is **uniformly convergent on** A_0. Sometimes we write

$$f_n \rightrightarrows f \text{ on } A_0, \quad \text{or} \quad f_n(x) \rightrightarrows f(x) \quad \text{for } x \in A_0.$$

Remark 4.1.2. If the sequence (f_n) is uniformly convergent on A_0 to f, then this sequence also converges pointwise on A_0 to f. That the converse is not always true.

It is sometimes useful to have the following necessary and sufficient condition for a sequence (f_n) to fail to converge uniformly on A_0 to f.

Lemma 4.1.3. *A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$* **does not converge uniformly** *on $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ if and only if for some $\varepsilon_0 > 0$ there is a subsequence (f_{n_k}) of (f_n) and a sequence (x_k) in A_0 such that*

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0 \quad \textit{for all} \quad k \in \mathbb{N}.$$

Proof. Negate the Definition 30: For some $\varepsilon_0 > 0$ there exists a sequence $|f_n(x) - f(x)| \geq \varepsilon$. Hence we can say that there exists a subsequence $(f_{n_k}(x_k))$ such that $|f_{n_k}(x_k) - x_k| \geq \varepsilon$ for all $k \in \mathbb{N}$. □

Example 4.1.3.

(a) Consider Example 4.1.1(a), $f_n(x) := x/n$ and $f(x) = 0$. If we let $n_k := k$ and $x_k := k$, then $f_{n_k}(x_k) = x_k/n_k = 1$ so that $|f_{n_k}(x_k) - f(x_k)| = |1 - 0| = 1$. Hence for $\varepsilon_0 = 1/2$ there exists a subsequence (x_k) such that $|f_{n_k}(x_k) - f(x_k)| > \varepsilon_0$. Therefore the sequence (f_n) does not converge uniformly on $\mathbb{R}$ to f.

(b) Consider Example 4.1.1(b), $g_n(x) = x^n$ and $g(x) = 0$ for $-1 < x < 1$. If $n_k := k$ and $x_k := \left(\frac{1}{2}\right)^{1/k}$,(so for any $k \in \mathbb{N}$ $0 < x_k < 1$. Hence $g(x_k) = 0$), then

$$|g_{n_k}(x_k) - g(x_k)| = |(x_k^{n_k}) - g(x_k)| = \left|\frac{1}{2} - 0\right| = \frac{1}{2}.$$

If we choose $\varepsilon_0 = 1/4$ we get the hypothesis of Lemma 4.1.3. Therefore the sequence (g_n) does not converge uniformly on $(-1, 1]$ to g.

(c) Consider Example 4.1.1(c), $h_n(x) = \dfrac{x^2 + nx}{n}$ and $h(x) = x$. If $n_k := k$ and $x_k := -k$, then $h_{n_k}(x_k) = 0$ and $h(x_k) = -k$ so that $|h_{n_k}(x_k) - h(x_k)| = k$. Therefore the sequence (h_n) does not converge uniformly on $\mathbb{R}$ to h.

The Uniform Norm

In discussing uniform convergence, it is often convenient to use the notion of the uniform norm on a set of bounded functions.

Definition 31. If $A \subseteq \mathbb{R}$ and $\varphi : A \to \mathbb{R}$ is a function, we say that φ is **bounded on** A if the set $\varphi(A)$ is a bounded subset of $\mathbb{R}$. If φ is bounded we define the **uniform norm** of φ on A by

$$\|\varphi\|_A := \sup\{|\varphi(x)| : x \in A\}.$$

Note that it follows that if $\varepsilon > 0$, then

$$\|\varphi\|_A \leq \varepsilon \quad \iff \quad |\varphi(x)| \leq \varepsilon \quad \text{for all} \quad x \in A$$

Lemma 4.1.4. *A sequence (f_n) of bounded functions on $A \subseteq \mathbb{R}$ converges uniformly on A to f if and only if $\|f_n - f\|_A \to 0$.*

Proof. ($\Rightarrow$) If (f_n) converges uniformly on A to f, then by Definition 30, given any $\varepsilon > 0$ there exists $K(\varepsilon)$ such that if $n \geq K(\varepsilon)$ and $x \in A$ then

$$|f_n(x) - f(x)| \leq \varepsilon.$$

From the definition of supremum, it follows that $\|f_n - f\|_A \leq \varepsilon$ whenever $n \geq K(\varepsilon)$. Since $\varepsilon > 0$ is arbitrary this implies that $\|f_n - f\|_A \to 0$

($\Leftarrow$) If $\|f_n - f\|_A \to 0$, then given $\varepsilon > 0$ there is a natural number $H(\varepsilon)$ such that if $n \geq H(\varepsilon)$ then $\|f_n - f\|_A \leq \varepsilon$. It follows from definition of uniform norm, that $|f_n(x) - f(x)| \leq \varepsilon$ for all $n \geq H(\varepsilon)$ and $x \in A$. Therefore (f_n) converges uniformly on A to f. $\square$

We now illustrate the use of Lemma 4.1.4 as a tool in examining a sequence of bounded functions for uniform convergence.

Example 4.1.4.

(a) We cannot apply Lemma 4.1.4 to the sequence in Example 4.1.1(a) since the function $f_n(x) - f(x) = x/n$ is not bounded on $\mathbb{R}$.

For the sake of illustration, let $A := [0,1]$. Although the sequence (x/n) did not converge uniformly on $\mathbb{R}$ to the zero function, we shall show that the convergence is uniform on A. To see this, we observe that

$$\|f_n - f\|_A = \sup\{|x/n - 0| : 0 \leq x \leq 1\} = \frac{1}{n}$$

so that $\|f_n - f\|_A \to 0$. Therefore (f_n) is uniformly convergent on A to f.

(b) Let $g_n(x) := x^n$ for $x \in A := [0,1]$ and $n \in \mathbb{N}$, and let $g(x) := 0$ for $0 \leq x < 1$ and $g(1) := 1$. The functions $g_n(x) - g(x)$ are bounded on A and

$$\|g_n - g\|_A = \sup \left\{ \begin{array}{lll} x^n & \text{for} & 0 \leq x < 1 \\ 0 & \text{for} \quad x = 1 & \end{array} \right\} = 1$$

for any $n \in \mathbb{N}$. Since $\|g_n - g\|_A$ does not converge to 0 , we infer that the sequence (g_n) does not converge uniformly on A to g.

(c) We cannot apply Lemma 4.1.4 to the sequence in Example 4.1.1(c) since the function $h_n(x) - h(x) = x^2/n$ is not bounded on $\mathbb{R}$.

Instead, let $A := [0,8]$ and consider

$$\|h_n - h\|_A = \sup\left\{x^2/n : 0 \le x \le 8\right\} = 64/n.$$

Therefore, the sequence (h_n) converges uniformly on A to h.

(d) If we refer to Example 4.1.1(d), we see that $|F_n - F|_{\mathbb{R}} \le 1/n$. Hence (F_n) converges uniformly on $\mathbb{R}$ to F.

(e) Let $G(x) := x^n(1-x)$ for $x \in A := [0,1]$. Then the sequence $(G_n(x))$ converges to $G(x) := 0$ for each $x \in A$. To calculate the uniform norm of $G_n - G = G_n$ on A, we find the derivative and solve

$$G_n'(x) = x^{n-1}(n - (n+1)x) = 0$$

to obtain the point $x_n := n/(n+1)$. This is an interior point of $[0,1]$, and it is easily verified by using the First Derivative Test, that G_n attains a maximum on $[0,1]$ at x_n. Therefore, we obtain

$$\|G_n\|_A = G_n\left(x_n\right) = (1+1/n)^{-n} \cdot \frac{1}{n+1}$$

which converges to $(1/e) \cdot 0 = 0$. Thus we see that convergence is uniform on A.

By making use of the uniform norm, we can obtain a **necessary and sufficient** condition for uniform convergence that is often useful.

Theorem 4.1.5. Cauchy Criterion for Uniform Convergence *Let (f_n) be a sequence of bounded functions on $A \subseteq \mathbb{R}$. Then this sequence converges uniformly on A to a bounded function f if and only if for each $\varepsilon > 0$ there is a number $H(\varepsilon)$ in $\mathbb{N}$ such that for all $m, n \ge H(\varepsilon)$, then $\|f_m - f_n\|_A \le \varepsilon$.*

Proof. ($\Rightarrow$) If $f_n \rightrightarrows f$ on A, then given $\varepsilon > 0$ there exists a natural number $K\left(\frac{1}{2}\varepsilon\right)$ such that if $n \geq K\left(\frac{1}{2}\varepsilon\right)$ then $\|f_n - f\|_A \leq \frac{1}{2}\varepsilon$. Hence, if both $m, n \geq K\left(\frac{1}{2}\varepsilon\right)$, then we conclude that

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f_n(x) - f(x)| \leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

for all $x \in A$. Therefore $\|f_m - f_n\|_A \leq \varepsilon$ for $m, n \geq K\left(\frac{1}{2}\varepsilon\right) =: H(\varepsilon)$.

($\Leftarrow$) Conversely, suppose that for $\varepsilon > 0$ there is $H(\varepsilon)$ such that if $m, n \geq H(\varepsilon)$, then $\|f_m - f_n\|_A \leq \varepsilon$. Therefore, for each $x \in A$ we have

$$|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_A \leq \varepsilon \quad \text{for} \quad m, n \geq H(\varepsilon). \tag{4.1}$$

It follows that $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$; therefore, by Theorem 1.2.6, it is a convergent sequence. We define $f : A \to \mathbb{R}$ by

$$f(x) := \lim\left(f_n(x)\right) \quad \text{for} \quad x \in A.$$

If we let $n \to \infty$ in (4.1), it follows from Theorem 1.2.3 that for each $x \in A$ we have

$$|f_m(x) - f(x)| \leq \varepsilon \quad \text{for} \quad m \geq H(\varepsilon).$$

Therefore the sequence (f_n) converges uniformly on A to f.

$\square$

EXERCISES

1. Show that $\lim(x/(x+n)) = 0$ for all $x \in \mathbb{R}, x \geq 0$.

2. Show that $\lim\left(nx/\left(1+n^2x^2\right)\right) = 0$ for all $x \in \mathbb{R}$.

3. Evaluate $\lim(nx/(1+nx)\ )$ for $x \in \mathbb{R}, x \geq 0$.

4. Evaluate $\lim\left(x^n/\left(1+x^n\right)\right)$ for $x \in \mathbb{R}, x \geq 0$.

5. Evaluate $\lim((\sin nx)/(1+nx))$ for $x \in \mathbb{R}, x \geq 0$.

6. Show that $\lim(\operatorname{Arctan} nx) = (\pi/2)\ \operatorname{sgn}\ x$ for $x \in \mathbb{R}$.

7. Evaluate $\lim\left(e^{-nx}\right)$ for $x \in \mathbb{R}, x \geq 0$.

8. Show that $\lim\left(xe^{-nx}\right) = 0$ for $x \in \mathbb{R}, x \geq 0$.

9. Show that $\lim\left(x^2e^{-nx}\right) = 0$ and that $\lim\left(n^2x^2e^{-nx}\right) = 0$ for $x \in \mathbb{R}, x \geq 0$.

10. Show that $\lim\left((\cos \pi x)^{2n}\right)$ exists for all $x \in \mathbb{R}$. What is its limit?

11. Show that if $a > 0$, then the convergence of the sequence in Exercise 1 is uniform on the interval $[0, a]$, but is not uniform on the interval $[0, \infty)$.

12. Show that if $a > 0$, then the convergence of the sequence in Exercise 2 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $[0, \infty)$

13. Show that if $a > 0$, then the convergence of the sequence in Exercise 3 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $[0, \infty)$.

14. Show that if $0 < b < 1$, then the convergence of the sequence in Exercise 4 is uniform on the interval $[0, b]$, but is not uniform on the interval $[0, 1]$.

15. Show that if $a > 0$, then the convergence of the sequence in Exercise 5 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $[0, \infty)$.

16. Show that if $a > 0$, then the convergence of the sequence in Exercise 6 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $(0, \infty)$

17. Show that if $a > 0$, then the convergence of the sequence in Exercise 7 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $[0, \infty)$.

18. Show that the convergence of the sequence in Exercise 8 is uniform on $[0, \infty\ )$.

19. Show that the sequence $\left(x^2 e^{-nx}\right)$ converges uniformly on $[0, \infty)$.

20. Show that if $a > 0$, then the sequence $\left(n^2 x^2 e^{-nx}\right)$ converges uniformly on the interval $[a, \infty)$, but that it does not converge uniformly on the interval $[0, \infty)$

21. Show that if $(f_n), (g_n)$ converge uniformly on the set A to f, g, respectively, then $(f_n + g_n)$ converges uniformly on A to $f + g$.

22. Show that if $f_n(x) := x + 1/n$ and $f(x) := x$ for $x \in \mathbb{R}$, then (f_n) converges uniformly on $\mathbb{R}$ to f, but the sequence $\left(f_n^2\right)$ does not converge uniformly on $\mathbb{R}$. (Thus the product of uniformly convergent sequences of functions may not converge uniformly.)

23. Let $(f_n), (g_n)$ be sequences of bounded functions on A that converge uniformly on A to f, g, respectively. Show that $(f_n g_n)$ converges uniformly on A to fg.

24. Let (f_n) be a sequence of functions that converges uniformly to f on A and that satisfies $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in A$. If g is continuous on the interval $[-M, M]$, show that the sequence $(g \circ f_n)$ converges uniformly to $g \circ f$ on A.

4.2 Interchange of Limits

It is often useful to know whether the limit of a sequence of functions is a continuous function, a differentiable function, or a Riemann integrable function. Unfortunately, it is not always the case that the limit of a sequence of functions possesses these useful properties.

Example 4.2.1.

(a) Let $g_n(x) := x^n$ for $x \in [0,1]$ and $n \in \mathbb{N}$. Then, the sequence (g_n) converges pointwise to the function

$$g(x) := \begin{cases} 0 & \text{for} \quad 0 \leq x < 1 \\ 1 & \text{for} \quad x = 1 \end{cases}$$

Although all of the functions g_n are continuous at $x = 1$, the limit function g is not continuous at $x = 1$. If we choose $x_k = \left(\frac{1}{2}\right)^{1/k}$ we get that this sequence does not converge uniformly to g on $[0,1]$.

(b) Each of the functions $g_n(x) = x^n$ in part (a) has a continuous derivative on $[0,1]$. However, the limit function g does not have a derivative at $x = 1$, since it is not continuous at that point.

(c) Let $f_n : [0,1] \to \mathbb{R}$ be defined for $n \geq 2$ by

$$f_n(x) := \begin{cases} n^2 x & \text{for} \quad 0 \leq x \leq 1/n \\ -n^2(x - 2/n) & \text{for} \quad 1/n \leq x \leq 2/n, \\ 0 & \text{for} \quad 2/n \leq x \leq 1 \end{cases}$$

(See Figure 4.3) It is clear that each of the functions f_n is continuous on $[0,1]$; hence it is Riemann integrable. Either by means of a direct calculation, or by referring to the significance of the integral as an area, we obtain

$$\int_0^1 f_n(x)dx = 1 \quad \text{for} \quad n \geq 2$$

The reader may show that $f_n(x) \to 0$ for all $x \in [0,1]$; hence the limit function f vanishes identically and is continuous (and hence integrable), and $\int_0^1 f(x)dx = 0$. Therefore we have the uncomfortable situation that:

$$\int_0^1 f(x)dx = 0 \neq 1 = \lim \int_0^1 f_n(x)dx$$

(d) Those who consider the functions f_n in part (c) to be "artificial" may prefer to consider the sequence (h_n) defined by $h_n(x) := 2nxe^{-nx^2}$ for $x \in [0,1], n \in \mathbb{N}$. Since $h_n = H_n'$, where $H_n(x) := -e^{-nx^2}$, the Fundamental Theorem 3.3.1 gives

$$\int_0^1 h_n(x)dx = H_n(1) - H_n(0) = 1 - e^{-n}.$$

For $x = 0$, $h_n(x) = 0$.

For $0 < x \leq 1$,
$\lim\limits_{n\to\infty} h_n(x) = \lim\limits_{n\to\infty} \frac{2nx}{e^{nx^2}} \qquad (= \frac{\infty}{\infty} \quad \text{form})$
$= \lim\limits_{n\to\infty} \frac{2x}{e^{nx^2}x^2} = \lim\limits_{n\to\infty} \frac{2}{e^{nx^2}x} = 0.$

ie, $h(x) := \lim\left(h_n(x)\right) = 0$ for all $x \in [0, 1]$; hence

$$\int_0^1 h(x)dx \neq \lim \int_0^1 h_n(x)dx$$

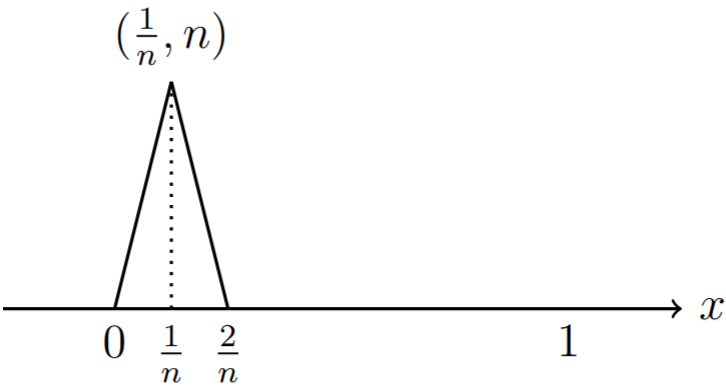

Figure 4.3: Example 4.2.1(c)

Although the extent of the discontinuity of the limit function in Example 4.2.1(a) is not very great, it is evident that more complicated examples can be constructed that will produce more extensive discontinuity. In any case, we must abandon the hope that the limit of a convergent sequence of continuous [respectively, differentiable, integrable] functions will be continuous [respectively, differentiable, integrable].

It will now be seen that the additional hypothesis of uniform convergence is sufficient to guarantee that the limit of a sequence of continuous functions is continuous. Similar results will also be established for sequences of differentiable and integrable functions.

Interchange of Limit and Continuity

Theorem 4.2.2. *Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that (f_n) converges uniformly on A to a function $f : A \to \mathbb{R}$. Then f is continuous on A.*

Proof. By hypothesis, given $\varepsilon > 0$ there exists a natural number $H := H\left(\frac{1}{3}\varepsilon\right)$ such that if $n \geq H$ then $|f_n(x) - f(x)| < \frac{1}{3}\varepsilon$ for all $x \in A$. Let $c \in A$ be arbitrary; we will show that f is continuous at c. By the Triangle Inequality we have

$$\begin{aligned} |f(x) - f(c)| &\leq |f(x) - f_H(x)| + |f_H(x) - f_H(c)| + |f_H(c) - f(c)| \\ &\leq \frac{1}{3}\varepsilon + |f_H(x) - f_H(c)| + \frac{1}{3}\varepsilon. \end{aligned}$$

Since f_H is continuous at c, there exists a number $\delta := \delta\left(\frac{1}{3}\varepsilon, c, f_H\right) > 0$ such that if $|x - c| < \delta$ and $x \in A$, then $|f_H(x) - f_H(c)| < \frac{1}{3}\varepsilon$. Therefore, if $|x - c| < \delta$ and $x \in A$, then we have $|f(x) - f(c)| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this establishes the continuity of f at the arbitrary point $c \in A$. (See Figure 4.4) $\square$

Remark 4.2.1. Although the uniform convergence of the sequence of continuous functions is sufficient to guarantee the continuity of the limit function, it is not necessary.

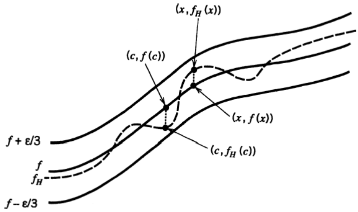

Figure 4.4: $|f(x) - f(c)| < \varepsilon$

Interchange of Limit and Derivative

Weierstrass showed that the function defined by the series

$$f(x) := \sum_{k=0}^{\infty} 2^{-k} \cos\left(3^k x\right)$$

is continuous at every point but does not have a derivative at any point in $\mathbb{R}$. By considering the partial sums of this series, we obtain a sequence of functions (f_n) that possess a derivative at every point and are uniformly convergent to f. Thus, even though the sequence of differentiable functions (f_n) is uniformly convergent, it does not follow that the limit function is differentiable. (See Exercises 9 and 10.)

We now show that if the sequence of derivatives (f_n') is uniformly convergent, then all is well. If one adds the hypothesis that the derivatives are continuous, then it is possible to give a short proof, based on the integral. (See Exercise 11.) However, if the derivatives are not assumed to be continuous, a somewhat more delicate argument is required.

Theorem 4.2.3. *Let $J \subseteq \mathbb{R}$ be a bounded interval and let (f_n) be a sequence of functions on J to $\mathbb{R}$. Suppose that there exists $x_0 \in J$ such that $(f_n(x_0))$ converges, and that the sequence (f_n') of derivatives exists* *on J and converges uniformly on J to a function g. Then the sequence (f_n) converges uniformly on J to a function f that has a derivative at every point of J and $f' = g$.*

Interchange of Limit and Integral

We have seen in Example 4.2.1(c) that if (f_n) is a sequence in $\mathcal{R}[a,b]$ that converges on $[a,b]$ to a function f in $\mathcal{R}[a,b]$, then it need not happen that

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n \tag{4.2}$$

We will now show that uniform convergence of the sequence is sufficient to guarantee that this equality holds.

Theorem 4.2.4. *Let (f_n) be a sequence of functions in $\mathcal{R}[a,b]$ and suppose that (f_n) converges* **uniformly** *on $[a,b]$ to f. Then $f \in \mathcal{R}[a,b]$ and (4.2) holds.*

Proof. It follows from the Cauchy Criterion 4.1.5 that given $\varepsilon > 0$ there exists $H(\varepsilon)$ such that if $m > n \geq H(\varepsilon)$ then

$$-\varepsilon \leq f_m(x) - f_n(x) \leq \varepsilon \quad \text{for} \quad x \in [a,b].$$

Properties of integral implies that

$$-\varepsilon(b-a) \leq \int_a^b f_m - \int_a^b f_n \leq \varepsilon(b-a).$$

Since $\varepsilon > 0$ is arbitrary, the sequence $\left(\int_a^b f_m\right)$ is a Cauchy sequence in $\mathbb{R}$ and therefore converges to some number, say $A \in \mathbb{R}$.

We now show $f \in \mathcal{R}[a,b]$ with integral A. If $\varepsilon > 0$ is given, let $K(\varepsilon)$ be such that if $m > K(\varepsilon)$, then $|f_m(x) - f(x)| < \varepsilon$ for all $x \in [a,b]$. If $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is any tagged partition of $[a,b]$ and if $m > K(\varepsilon)$, then

$$\begin{aligned}
\left|S\left(f_m;\dot{\mathcal{P}}\right) - S(f;\dot{\mathcal{P}})\right| &= \left|\sum_{i=1}^n \{f_m(t_i) - f(t_i)\}(x_i - x_{i-1})\right| \\
&\le \sum_{i=1}^n |f_m(t_i) - f(t_i)|(x_i - x_{i-1}) \\
&\le \sum_{i=1}^n \varepsilon(x_i - x_{i-1}) = \varepsilon(b-a).
\end{aligned}$$

We now choose $r \ge K(\varepsilon)$ such that $\left|\int_a^b f_r - A\right| < \varepsilon$ and we let $\delta_{r,\varepsilon} > 0$ be such that $\left|\int_\alpha^b f_r - S\left(f_r;\dot{\mathcal{P}}\right)\right| < \varepsilon$ whenever $\|\dot{\mathcal{P}}\| < \delta_{r,c}$. Then we have

$$\begin{aligned}
|S(f;\dot{\mathcal{P}}) - A| \le{}& \left|S(f;\dot{\mathcal{P}}) - S\left(f_r;\dot{\mathcal{P}}\right)\right| + \left|S\left(f_r;\dot{\mathcal{P}}\right) - \int_a^b f_r\right| + \\
& \left|\int_a^b f_r - A\right| \\
\le{}& \varepsilon(b-a) + \varepsilon + \varepsilon = \varepsilon(b-a+2)
\end{aligned}$$

But since $\varepsilon > 0$ is arbitrary, it follows that $f \in \mathcal{R}[a,b]$ and $\int_a^b f = A$.

□

The hypothesis of uniform convergence is a very stringent one and restricts the utility of this result. For the present, we will state a result that does not require the uniformity of the convergence, but does require that the limit function be Riemann integrable.

Theorem 4.2.5. Bounded Convergence Theorem *Let (f_n) be a sequence in $\mathcal{R}[a,b]$ that converges on $[a,b]$ to a function $f \in \mathcal{R}[a,b]$. Suppose also that there exists $B > 0$ such that $|f_n(x)| \leq B$ for all $x \in [a,b], n \in \mathbb{N}$. Then equation (4.2) holds.*

EXERCISES

1. Show that the sequence $(x^n/(1+x^n))$ does not converge uniformly on $[0,2]$ by showing that the limit function is not continuous on $[0,2]$.

2. Prove that the sequence in Example 4.2.1(c) is an example of a sequence of continuous functions that converges nonuniformly to a continuous limit.

3. Construct a sequence of functions on $[0,1]$ each of which is discontinuous at every point of $[0,1]$ and which converges uniformly to a function that is continuous at every point.

4. Suppose (f_n) is a sequence of continuous functions on an interval I that converges uniformly on I to a function f. If $(x_n) \subseteq I$ converges to $x_0 \in I$, show that $\lim (f_n(x_n)) = f(x_0)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$ and let $f_n(x) := f(x + 1/n)$ for $x \in \mathbb{R}$. Show that (f_n) converges uniformly on $\mathbb{R}$ to f.

6. Let $f_n(x) := 1/(1+x)^n$ for $x \in [0,1]$. Find the pointwise limit f of the sequence (f_n) on $[0,1]$. Does (f_n) converge uniformly to f on $[0,1]$?

7. Suppose the sequence (f_n) converges uniformly to f on the set A, and suppose that each f_n is bounded on A. (That is, for each n there is a constant M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$. Show that the function f is bounded on A

8. Let $f_n(x) := nx/\left(1+nx^2\right)$ for $x \in A := [0,\infty)$. Show that each f_n is bounded on A, but the pointwise limit f of the sequence is not bounded on A. Does (f_n) converge uniformly to f on A ?

9. Let $f_n(x) := x^n/n$ for $x \in [0,1]$. Show that the sequence (f_n) of differentiable functions converges uniformly to a differentiable function f on $[0,1]$, and that the sequence (f_n') converges on $[0,1]$ to a function g, but that $g(1) \neq f'(1)$.

10. Let $g_n(x) := e^{-nx}/n$ for $x \geq 0, n \in \mathbb{N}$. Examine the relation between $\lim(g_n)$ and $\lim(g_n')$.

11. Let $I := [a,b]$ and let (f_n) be a sequence of functions on $I \to \mathbb{R}$ that converges on I to f. Suppose that each derivative f_n' is continuous on I and that the sequence (f_n') is uniformly convergent to g on I. Prove that $f(x) - f(a) = \int_a^x g(t)dt$ and that $f'(x) = g(x)$ for all $x \in I$.

12. Show that $\lim \int_1^2 e^{-nx^2} dx = 0$.

13. If $a > 0$, show that $\lim \int_a^\pi (\sin nx)/(nx)dx = 0$. What happens if $a = 0$?

14. Let $f_n(x) := nx/(1+nx)$ for $x \in [0,1]$. Show that (f_n) converges nonuniformly to an integrable function f and that $\int_0^1 f(x)dx = \lim \int_0^1 f_n(x)dx$.

15. Let $g_n(x) := nx(1-x)^n$ for $x \in [0,1], n \in \mathbb{N}$. Discuss the convergence of (g_n) and $\left(\int_0^1 g_n dx\right)$.

16. Let $\{r_1, r_2, \ldots, r_{\text{n}} \ldots\}$ be an enumeration of the rational numbers in $I := [0,1]$, and let $f_n : I \to \mathbb{R}$ be defined to be 1 if $x = r_1, \ldots, r_n$ and equal to 0 otherwise. Show that f_n is Riemann integrable for each $n \in \mathbb{N}$, that $f_1(x) \leq f_2(x) \leq \cdots \leq f_n(x) \leq \cdots$, and that $f(x) := \lim (f_n(x))$ is the Dirichlet function, which is not Riemann integrable on $[0,1]$.

17. Let $f_n(x) := 1$ for $x \in (0, 1/n)$ and $f_n(x) := 0$ elsewhere in $[0,1]$. Show that (f_n) is a decreasing sequence of discontinuous functions that converges to a continuous limit function, but the convergence is not uniform on $[0,1]$.

18. Let $f_n(x) := x^n$ for $x \in [0,1], n \in \mathbb{N}$. Show that (f_n) is a decreasing sequence of continuous functions that converges to a function that is not continuous, but the convergence is not uniform on $[0,1]$.

19. Let $f_n(x) := x/n$ for $x \in [0,\infty), n \in \mathbb{N}$. Show that (f_n) is a decreasing sequence of continuous functionsthat converges to a continuous limit function, but the convergence is not uniform on $[0,\infty)$

20. Give an example of a decreasing sequence (f_n) of continuous functions on $[0,1)$ that converges to a continuous limit function, but the convergence is not uniform on $[0,1)$.

4.3 Series of Functions

Because of their frequent appearance and importance, we now present a discussion of infinite series of functions. Since the convergence of an infinite series is handled by examining the sequence of partial sums, questions concerning series of functions are answered by examining corresponding questions for sequences of functions. For this reason, a portion of the present section is merely a translation of facts already established for sequences of functions into series terminology.

Definition 32. If (f_n) is a sequence of functions defined on a subset D of $\mathbb{R}$ with values in $\mathbb{R}$, the sequence of **partial sums** (s_n) of the infinite series $\sum f_n$ is defined for x in D by

$$
\begin{aligned}
s_1(x) &:= f_1(x) \\
s_2(x) &:= s_1(x) + f_2(x) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
s_{n+1}(x) &:= s_n(x) + f_{n+1}(x) \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

In case the sequence (s_n) of functions converges on D to a function f, we say that the infinite series of functions $\sum f_n$ **converges** to f on D. We will often write

$$\sum f_n \quad \text{or} \quad \sum_{n=1}^{\infty} f_n$$

to denote either the series or the limit function, when it exists.

If the series $\sum |f_n(x)|$ converges for each x in D, we say that $\sum f_n$ is **absolutely convergent** on D. If the sequence (s_n) of partial sums is uniformly convergent on D to f, we say that $\sum f_n$ is **uniformly**

convergent on D, or that it **converges to** f **uniformly** on D.

Theorem 4.3.1. *If f_n is continuous on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ for each $n \in \mathbb{N}$ and if $\sum f_n$ converges to f uniformly on D, then f is continuous on D.*

Theorem 4.3.2. *Suppose that the real-valued functions $f_n, n \in \mathbb{N}$, are Riemann integrable on the interval $J := [a, b]$. If the series $\sum f_n$ converges to f uniformly on J, then f is Riemann integrable and*

$$\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$$

Theorem 4.3.3. *For each $n \in \mathbb{N}$, let f_n be a real-valued junction on $J := [a, b]$ that has a derivative f_n' on J. Suppose that the series $\sum f_n$ converges for at least one point of J and that the series of derivatives $\sum f_n'$ converges uniformly on J. Then there exists a real-valued function f on J such that $\sum f_n$ converges uniformly on J to f. In addition, f has a derivative on J and $f' = \sum f_n'$.*

Tests for Uniform Convergence

Since we have stated some consequences of uniform convergence of series, we shall now present a few tests that can be used to establish uniform convergence.

Theorem 4.3.4. Cauchy Criterion *Let (f_n) be a sequence of functions on $D \subseteq \mathbb{R}$ to $\mathbb{R}$. The series $\sum f_n$ is uniformly convergent on D if and only if for every $\varepsilon > 0$ there exists an $M(\varepsilon)$ such that if $m > n \geq M(\varepsilon)$, then*

$$|f_{n+1}(x) + \cdots + f_m(x)| < \varepsilon \text{ for all } \quad x \in D.$$

Theorem 4.3.5. Weierstrass M – Test *Let (M_n) be a sequence of* *positive real numbers such that* $|f_n(x)| \leq M_n$ *for* $x \in D, n \in \mathbb{N}$. *If the series* $\sum M_n$ *is convergent, then* $\sum f_n$ *is uniformly convergent on* D.

Proof. If $m > n$, we have the relation

$$|f_{n+1}(x) + \cdots + f_m(x)| \leq M_{n+1} + \cdots + M_m \text{ for } \quad x \in D.$$

By using Cauchy Criterion for Series for $\sum M_n$ we get that, for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that if $m > n \geq N(\varepsilon)$, then

$$|M_{n+1} + \cdots + M_m| < \varepsilon$$

which implies that $|f_{n+1}(x) + \cdots + f_m(x)| < \varepsilon$. Hence by Cauchy criterion $\sum f_n$ is uniformly convergent on D.

□

EXERCISES

1. Let $\sum a_n$ be a given series and let $\sum b_n$ be the series in which the terms are the same and in the same order as in $\sum a_n$ except that the terms for which $a_n = 0$ have been omitted. Show that $\sum a_n$ converges to A if and only if $\sum b_n$ converges to A.

2. Show that the convergence of a series is not affected by changing a finite number of its terms. (Of course, the value of the sum may be changed.)

3. By using partial fractions, show that

   (a) $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1$,

(b) $\sum_{n=0}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha} > 0$, if $\alpha > 0$.

(c) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$.

4. If $\sum x_n$ and $\sum y_n$ are convergent, show that $\sum (x_n + y_n)$ is convergent.

5. Can you give an example of a convergent series $\sum x_n$ and a divergent series $\sum y_n$ such that $\sum (x_n + y_n)$ is convergent? Explain.

6. (a) Calculate the value of $\sum_{n=2}^{\infty}(2/7)^n$. (Note the series starts at $n = 2$.)

   (b) Calculate the value of $\sum_{n=1}^{\infty}(1/3)^{2n}$. (Note the series starts at $n = 1$.)

7. Find a formula for the series $\sum_{n=1}^{\infty} r^{2n}$ when $|r| < 1$.

8. Let $r_1, r_2, \ldots, r_n, \ldots$ be an enumeration of the rational numbers in the interval $[0, 1]$. (See Section 1.3.) For a given $\varepsilon > 0$, put an interval of length ε^n about the nth rational number r_n for $n = 1, 2, 3, \ldots$, and find the total sum of the lengths of all the intervals. Evaluate this number for $\varepsilon = 0.1$ and $\varepsilon = 0.01$.

9. (a) Show that the series $\sum_{n=1}^{\infty} \cos n$ is divergent.

   (b) Show that the series $\sum_{n=1}^{\infty}(\cos n)/n^2$ is convergent. Use an argument similar to that in Example 3.7.6 (f) to show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is

10. If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum a_n^2$ always convergent? Either prove it or give a counterexample.

11. If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum \sqrt{a_n}$ always convergent? Either prove it or give a counterexample.

12. If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum \sqrt{a_n a_{n+1}}$ always convergent? Either prove it or give a counterexample.

13. If $\sum a_n$ with $a_n > 0$ is convergent, and if $b_n := (a_1 + \cdots + a_n)/n$ for $n \in \mathbb{N}$, then show that $\sum b_n$ is always divergent.

14. Let $\sum_{n=1}^{\infty} a(n)$ be such that $(a(n))$ is a decreasing sequence of strictly positive numbers. If $s(n)$ denotes the nth partial sum, show (by grouping the terms in $s(2^n)$ in two different ways) that $\frac{1}{2}\left(a(1) + 2a(2) + \cdots + 2^n a(2^n)\right) \le s(2^n) \le \left(a(1) + 2a(2) + \cdots + 2^{n-1} a\left(2^{n-1}\right)\right) + a(2^n)$.

    Use these inequalities to show that $\sum_{n=1}^{\infty} a(n)$ converges if and only if $\sum_{n=1}^{\infty} 2^n a(2^n)$ converges. This result is often called the Cauchy Condensation Test; it is very powerful.

15. Use the Cauchy Condensation Test to discuss the p-series $\sum_{n=1}^{\infty} (1/n^p)$ for $p > 0$.

16. Use the Cauchy Condensation Test to establish the divergence of the series:

    (a) $\sum \frac{1}{n \ln n}$,

    (b) $\displaystyle\sum \frac{1}{n(\ln n)(\ln\ln n)}$,

    (c) $\displaystyle\sum \frac{1}{n(\ln n)(\ln\ln n)(\ln\ln\ln n)}$.

17. Show that if $c > 1$, then the following series are convergent:

    (a) $\displaystyle\sum \frac{1}{n(\ln n)^c}$,

    (b) $\displaystyle\sum \frac{1}{n(\ln n)(\ln\ln n)^c}$.

18. Discuss the convergence and the uniform convergence of the series $\sum f_n$, where $f_n(x)$ is given by:

(a) $\left(x^2+n^2\right)^{-1}$,

(b) $(nx)^{-2} \quad (x \neq 0)$,

(c) $\sin\left(x/n^2\right)$

(d) $(x^n+1)^{-1} \quad (x \neq 0)$,

(e) $x^n/\left(x^n+1\right) \quad (x \geq 0)$,

(f) $(-1)^n(n+x)^{-1}$ $(x \geq 0)$

19. If $\sum a_n$ is an absolutely convergent series, then the series $\sum a_n \sin nx$ is absolutely and uniformly convergent.

20. Let (c_n) be a decreasing sequence of positive numbers. If $\sum c_n \sin nx$ is uniformly convergent, then $\lim\left(nc_n\right)=0$.

Chapter 5

IMPROPER RIEMANN INTEGRALS

5.1 Definitions and Examples

Many theorems in Mathematics and many applications in science and technology depend on the evaluation and on the properties of improper Riemann integrals. Therefore, we are going to state the definitions of improper or generalized integrals and then discuss their properties. Subsequently, we discuss criteria for checking their existence (or non-existence) and then we develop methods and mathematical techniques we can use in order to evaluate them. Certainly the answers to many important improper Riemann integrals have been tabulated in mathematical handbooks and can also be found with the help of various computer programs, which we can use if we can trust in them, of course. However, these means can never exhaust every interesting case. Hence, the good knowledge of the mathematical theory of how to understand, handle and compute improper integrals, at a higher level, will always remain very important for being able to deal with new cases and checking the accuracy of the answers provided in tables or found by computer programs or packages.

In a regular undergraduate Calculus course we study the Fundamental Theorem of Integral Calculus. This states:

Theorem 5.1.1. *(**Fundamental Theorem of Integral Calculus**) If a real function*
$f : [a,b] \to \mathbb{R}$ ($a < b$ are real numbers) is continuous, then it possesses antiderivatives $F(x)$, i.e., functions that satisfy $F'(x) = f(x)$ for every $x \in [a,b]$ [at the end points we consider the appropriate side derivatives, $F'_+(a) = f(a)$ and $F'_-(b) = f(b)$]. Any such antiderivative $F(x)$ of $f(x)$ is necessarily continuous in (a,b), right continuous at a, [i.e., $F(a) = \lim_{x\to a^+} F(x)$], left continuous at b, [i.e., $F(b) = \lim_{x\to b^-} F(x)$], and satisfies

$$\int_a^b f(x)dx = F(b) - F(a).$$

We emphasize the three hypotheses that must hold in order for this Theorem to be valid:

1. $[a,b]$ is a closed and bounded interval in the real line.
2. $f(x)$ is continuous and therefore, by the extreme value theorem, bounded on $[a,b]$.
3. In the computation $F(b) - F(a), F(x)$ can be any fixed continuous antiderivative.

We know that on $[a,b]$, there are infinitely many antiderivatives of a continuous function $f(x)$, but any two of them differ by a real constant C. Since they are differentiable they are continuous, and since their derivative is the continuous function $f(x)$ they are continuously differentiable.

Under these conditions the integral

$$\int_a^b f(x)dx = F(b) - F(a)$$

is called the **proper Riemann integral** of $f(x)$ over the interval $[a, b]$. This is well defined and equal to the limit of the **Riemann sums** of $f(x)$ over $[a, b]$, as the maximum length of the subintervals into which we subdivide $[a, b]$, in this well-known process, approaches zero. i.e.,

$$\int_a^b f(x)dx = \lim_{\max(\Delta x_k)\to 0} \sum_{k=1}^{n} f\left(x_k^*\right) \Delta x_k = F(b) - F(a).$$

We can go a bit beyond the undergraduate interpretation of the Fundamental Theorem of Integral Calculus and relax the above hypotheses as follows:

We more generally consider $f : [a, b] \to \mathbb{R}$ **piecewise continuous** and **bounded**. Then its Riemann integral exists. In such a case, we can also find $F(x)$ antiderivative of $f(x)$ which is continuous in (a, b), right continuous at a, left continuous at b and differentiable only at the points of continuity of $f(x)$. At the points of discontinuity of $f(x), F(x)$ may have a left or right derivative but not derivative.

Sometimes $f : [a, b] \to \mathbb{R}$ may be continuous in (a, b), right continuous at a, left continuous at b, but in order to obtain, by means of the usual methods and rules of antidifferentiation, an antiderivative $F(x)$ of $f(x)$ which is continuous in (a, b), right continuous at a, left continuous at b, we may have to make necessary adjustments, by adjusting certain constants, at certain points of the interval of integration $[a, b]$. Only then we can guarantee the **result**

$$\int_a^b f(x)dx = F(b) - F(a)$$

in such cases.

In fact, the Fundamental Theorem of Calculus proves that if $f(x)$ is a bounded Riemann integrable function on the closed and bounded interval $[a, b]$, then

$$F(x) = \int_a^x f(t)dt \text{ is continuous in [a, b]}$$

and satisfies

$$\frac{d}{dx}F(x)\bigg|_{x=w} = f(w) \text{ at all points of continuity } w \text{ of } f(x) \text{ on } [a, b]$$

So, if $f(x)$ is continuous on $[a, b]$, the function $F(x)$ is an antiderivative of $f(x)$ on $[a, b]$ and is continuously differentiable.

The anomaly we discuss here is not due to any deficiency of the Fundamental Theorem of Calculus, but it is created by the standard rules and methods of antidifferentiation. At times, the answers obtained by these rules are not defined at certain points and therefore are discontinuous at these points. To obtain the continuity as the Fundamental Theorem of Calculus guarantees and requires, we must adjust these answers appropriately. To understand this extraordinary situation and be aware of its occurrence, let us study the following example:

Example 5.1.2. We consider the function

$$f(x) = \frac{3}{5 - 4\cos(x)}.$$

This function is defined for every $x \in \mathbb{R}$. It is continuous at every $x \in \mathbb{R}$, bounded $\left[\frac{1}{3} \leq f(x) \leq 3\right]$, periodic with period 2π and even. We have

$$\cos(x) = \frac{1 - \tan^2\left(\dfrac{x}{2}\right)}{\tan^2\left(\dfrac{x}{2}\right) + 1}$$

Hence

$$\begin{aligned}
\int \frac{3}{5 - 4\cos(x)} dx &= \int \frac{3}{5 - 4\dfrac{1 - \tan^2\left(\dfrac{x}{2}\right)}{\tan^2\left(\dfrac{x}{2}\right) + 1}} dx \\
&= 3\int \frac{\tan^2\left(\dfrac{x}{2}\right) + 1}{5\tan^2\left(\dfrac{x}{2}\right) + 5 - 4 + 4\tan^2\left(\dfrac{x}{2}\right)} dx \\
&= 3\int \frac{\tan^2\left(\dfrac{x}{2}\right) + 1}{9\tan^2\left(\dfrac{x}{2}\right) + 1} dx
\end{aligned}$$

Substitute $u = \tan\left(\frac{x}{2}\right)$. Then $x = 2\arctan(u)$ and $dx = \dfrac{2}{u^2 + 1} du$

$$\begin{aligned}
\Rightarrow \int \frac{3}{5 - 4\cos(x)} dx &= 3\int \frac{u^2 + 1}{9u^2 + 1} \frac{2}{u^2 + 1} du \\
&= 3\int \frac{2}{9u^2 + 1} du = 3\int \frac{2}{(3u)^2 + 1} du
\end{aligned}$$

Put $3u = v$ then $3du = dv$.

$$\text{So} \quad \int \frac{3}{5-4\cos(x)}dx = 2\int \frac{dv}{v^2+1} = 2\arctan(v) + C$$
$$= 2\arctan(3u) + C$$
$$= 2\arctan\left(3\tan\left(\frac{x}{2}\right)\right) + C.$$

We let $C = 0$ (as we usually do in calculus when we evaluate definite integrals). So, we choose

$$F(x) = 2\arctan\left[3\tan\left(\frac{x}{2}\right)\right].$$

This function is defined for all real $x \neq (2k+1)\pi$, with $k \in \mathbb{Z}$, since at $x = (2k+1)\pi$, with $k \in \mathbb{Z}, \tan\left(\frac{x}{2}\right)$ is not defined. At these exceptional points, however, we have

$$\lim_{x \to (2k+1)\pi^-} F(x) = 2 \cdot \frac{\pi}{2} = \pi \quad \text{and} \quad \lim_{x \to (2k+1)\pi^+} F(x) = 2 \cdot \frac{-\pi}{2} = -\pi.$$

So, at each $x = (2k+1)\pi$, with $k \in \mathbb{Z}, F(x)$ has a jump discontinuity with jump $|\pi - (-\pi)| = 2\pi$. Notice also that $F(x)$ is bounded, $[-\pi < F(x) < \pi]$.

Therefore, to evaluate the definite integrals

$$\int_a^b \frac{3}{5-4\cos(x)}dx = F(b) - F(a), \qquad \text{for any} \qquad -\pi \leq a, b \leq \pi$$

we can use the continuous antiderivative

$$\bar{F}(x) = \begin{cases} -\pi, & \text{if} \quad x = -\pi^+ \\ 2\arctan\left[3\tan\left(\frac{x}{2}\right)\right], & \text{if} \quad -\pi < x < \pi \\ \pi, & \text{if} \quad x = \pi^-. \end{cases}$$

For example,

$$\int_{-\pi}^{\pi} \frac{3}{5-4\cos(x)} dx = \bar{F}\left(\pi^-\right) - \bar{F}\left(-\pi^+\right) = \pi - (-\pi) = 2\pi$$

But, we cannot use $F(x)$ or $\bar{F}(x)$ to evaluate definite integrals if a or b does not satisfy $-\pi \leq a, b \leq \pi$. For instance, if we use it with $a = -2\pi$ and $b = 4\pi$, we find

$$\int_{-2\pi}^{4\pi} \frac{3}{5-4\cos(x)} dx = F(4\pi) - F(-2\pi) = 0 - 0 = 0$$

which is incorrect, since the continuous function $f(x) > 0$, and therefore this definite integral should be > 0.

This error has occurred because the chosen antiderivative over the interval $[-2\pi, 4\pi]$ is discontinuous at the exceptional points examined above.

The correct answer is obtained if we use the adjusted antiderivative

$$F_c(x) = \begin{cases} 2\arctan\left[3\tan\left(\frac{x}{2}\right)\right] + 2\pi\left[\left[\frac{x+\pi}{2\pi}\right]\right], & \text{if } x \neq (2k+1)\pi, k \in \mathbb{Z} \\ (2k+1)\pi, & \text{if } x = (2k+1)\pi, k \in \mathbb{Z}, \end{cases}$$

where $\left[\left[\dfrac{x+\pi}{2\pi}\right]\right]$ is the integer part or floor function of $\dfrac{x+\pi}{2\pi}$.

This new $F_c(x)$ is now continuous, differentiable and $F_c'(x) = f(x)$ at

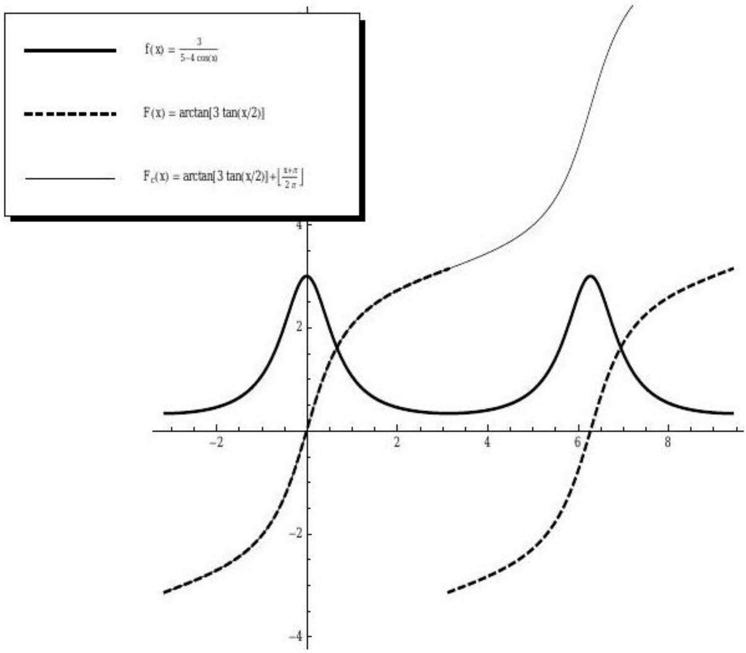

Figure 5.1: Three functions in Example 5.1.2

all points of $[-2\pi, 4\pi]$ as the Fundamental Theorem of Calculus claims and requires. (In fact, this is true at every $x \in \mathbb{R}$.)

Now, we get:

$$\int_{-2\pi}^{4\pi} \frac{3}{5 - 4\cos(x)} dx = F_c(4\pi) - F_c(-2\pi) = 4\pi - (-2\pi) = 6\pi > 0$$

This result is the correct one and was also expected since $f(x)$ is 2π-periodic, with integral 2π over $[-\pi, \pi]$ and we have integrated it over an interval of length 6π, i.e., three times its period.

Now we continue with the improper or generalized Riemann integrals.

Definition 33. An integral of a piecewise continuous real function of a real variable is called an **improper** or **generalized Riemann integral** if at least one of the following three conditions occurs:

1. The integrated function is unbounded over the interval of integration
2. The interval of integration is not closed.
3. The interval of integration is unbounded.

In all the pertinent definitions that follow, an improper or generalized Riemann integral of a real piecewise continuous function of a real variable defined over a set $I \subseteq \mathbb{R}$ will be defined to be a certain limit of proper Riemann integrals.

More concretely, we present four cases and definitions in our exposition, each of which may include two or more subcases, that generalize the proper Riemann integrals:

Definition 34. Suppose $y = f(x)$ is a real function continuous in $[a, b) \subset \mathbb{R}$, then we define:

For $b < \infty$

$$\int_a^b f(x)dx = \lim_{\rho \to b^-} \int_a^\rho f(x)dx = \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)dx$$

For $b = \infty$

$$\int_a^\infty f(x)dx = \lim_{M \to \infty} \int_a^M f(x)dx$$

Example 5.1.3.

1.

$$\begin{aligned}
\int_{-1}^{0} \frac{dx}{\sqrt[3]{x}} &= \lim_{\rho \to 0^-} \int_{-1}^{\rho} x^{\frac{-1}{3}}\, dx \\
&= \lim_{\rho \to 0^-} \left[\frac{3}{2} x^{\frac{2}{3}}\right]_{-1}^{\rho} \\
&= \lim_{\rho \to 0^-} \left[\frac{3}{2}\rho^{\frac{2}{3}} - \frac{3}{2}(-1)^{\frac{2}{3}}\right] \\
&= \frac{3}{2} \lim_{\rho \to 0^-} \rho^{\frac{2}{3}} - \frac{3}{2} \\
&= \frac{3}{2} \cdot 0 - \frac{3}{2} = -\frac{3}{2}
\end{aligned}$$

2.

$$\begin{aligned}
\int_{1}^{\infty} \frac{dx}{x^2+1} &= \lim_{M \to \infty} [\arctan(x)]_1^M \\
&= \lim_{M \to \infty} [\arctan(M) - \arctan(1)) \\
&= \lim_{M \to \infty} \arctan(M) - \frac{\pi}{4} \\
&= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.
\end{aligned}$$

3.

$$\begin{aligned}
\int_{1}^{\infty} \frac{dx}{\sqrt{x}} &= \lim_{M \to \infty} [2\sqrt{x}]_1^M \\
&= \lim_{M \to \infty} 2\sqrt{M} - 2\sqrt{1} \\
&= \infty - 2 = \infty
\end{aligned}$$

4.

$$\begin{aligned}\int_0^{\infty} \sin(x)dx &= \lim_{M\to\infty} [-\cos(x)]_0^M \\ &= \lim_{M\to\infty} [-\cos(M) + \cos(0)] \\ &= -\lim_{M\to\infty} \cos(M) + 1\end{aligned}$$

This limit does not exist, since $\cos(M)$ oscillates between -1 and 1.

Definition 35. Suppose $y = f(x)$ is a real continuous function in $(a, b] \subset \mathbb{R}$. Then we define:
For $-\infty < a$

$$\int_a^b f(x)dx = \lim_{\sigma\to a^+} \int_\sigma^b f(x)dx = \lim_{\delta\to 0^+} \int_{a+\delta}^b f(x)dx$$

For $a = -\infty$

$$\int_{-\infty}^b f(x)dx = \lim_{N\to-\infty} \int_N^b f(x)dx$$

Example 5.1.4.

1. $\displaystyle\int_0^2 \frac{dx}{\sqrt{x}} = \lim_{\sigma\to 0^+} [2\sqrt{x}]_\sigma^2$
   $\displaystyle= 2\sqrt{2} - \lim_{\sigma\to 0^+} 2\sqrt{\sigma} = 2\sqrt{2} - 0 = 2\sqrt{2}$

2. $\displaystyle\int_{-\infty}^0 e^x dx = \lim_{N\to-\infty} [e^x]_N^0$
   $\displaystyle= \lim_{N\to-\infty} \left[e^0 - e^N\right] = 1 - \lim_{N\to-\infty} e^N = 1 - 0 = 1$

3. $\displaystyle\int_{-\infty}^1 x^2 dx = \lim_{N\to-\infty} \left[\frac{x^3}{3}\right]_N^1 = \lim_{N\to-\infty} \left[\frac{1}{3} - \frac{N^3}{3}\right]$

$$= \frac{1}{3} - \lim_{N \to -\infty} \frac{N^3}{3} = \frac{1}{3} - (-\infty) = \infty.$$

Definition 36. Suppose $y = f(x)$ is a real continuous function in $(a, b) \subseteq \mathbb{R}$. Then we define:

For $-\infty < a < b < \infty$

$$\int_a^b f(x)dx = \lim_{\substack{\rho \to b^- \\ \sigma \to a^+}} \int_\sigma^\rho f(x)dx.$$

For $a = -\infty$ and $b = \infty$

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{\substack{M \to \infty \\ N \to -\infty}} \int_N^M f(x)dx.$$

For $a = -\infty$ and $b < \infty$

$$\int_{-\infty}^{b} f(x)dx = \lim_{\substack{\rho \to b^- \\ N \to -\infty}} \int_N^\rho f(x)dx.$$

For $-\infty < a$ and $b = \infty$

$$\int_a^{\infty} f(x)dx = \lim_{\substack{M \to \infty \\ \sigma \to a^+}} \int_\sigma^M f(x)dx.$$

In the above double limits, the two limiting processes are independent of each other in general.

Example 5.1.5. 1.

$$\begin{aligned}
\int_{-1}^{1} \frac{dx}{x^2-1} &= \int_{-1}^{1} \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx \\
&= \lim_{\substack{\rho\to 1^-\\ \sigma\to -1^+}} \int_{\sigma}^{\rho} \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx \\
&= \frac{1}{2} \lim_{\substack{\rho\to 1^-\\ \sigma\to -1^+}} \left[\ln|x-1| - \ln|x+1|\right]_{\sigma}^{\rho} \\
&= \frac{1}{2} \lim_{\substack{\rho\to 1^-\\ \sigma\to -1^+}} [\ln|\rho-1| - \\
&\quad \ln|\rho+1| - \ln|\sigma-1| + \ln|\sigma+1|]
\end{aligned}$$

We have that

$$\begin{array}{ll}
\lim_{\rho\to 1^-} \ln|\rho-1| = -\infty, & \lim_{\rho\to 1^-} \ln|\rho+1| = \ln(2) \\
\lim_{\sigma\to -1^+} \ln|\sigma-1| = \ln(2), & \lim_{\sigma\to -1^+} \ln|\sigma+1| = -\infty
\end{array}$$

So, the above improper integral as double limit is

$$\frac{1}{2}[-\infty - \ln(2) - \ln(2) - \infty] = -\infty.$$

2.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{x^2+1} &= \lim_{\substack{M\to\infty\\ N\to-\infty}} [\arctan(x)]_N^M \\
&= \lim_{\substack{M\to\infty\\ N\to-\infty}} [\arctan(M) - \arctan(N)] \\
&\qquad \text{(both partial limits exist separately)} \\
&= \lim_{M\to\infty} \arctan(M) - \lim_{N\to-\infty} \arctan(N)
\end{aligned}$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$
$$= \pi.$$

3.

$$\int_0^\infty \frac{dx}{\sqrt{x}} = \lim_{\substack{M\to\infty \\ \sigma\to 0^+}} [2\sqrt{x}]_\sigma^M = \lim_{M\to\infty} 2\sqrt{M} - \lim_{\sigma\to 0^+} 2\sqrt{\sigma}$$
$$= \infty - 0 = \infty$$

4.

$$\int_{-\infty}^0 \frac{dx}{x^2} = \lim_{\substack{\rho\to 0^- \\ N\to -\infty}} \left[\frac{-1}{x}\right]_N^\rho$$
$$= \lim_{\rho\to 0^-} \left(-\frac{1}{\rho}\right) - \lim_{N\to-\infty} \left(\frac{-1}{N}\right)$$
$$= -(-\infty) - 0 = \infty$$

5.

$$\int_{-\infty}^\infty x dx = \lim_{\substack{M\to\infty \\ N\to -\infty}} \left[\frac{x^2}{2}\right]_N^M$$
$$= \lim_{\substack{M\to\infty \\ N\to -\infty}} \left[\frac{M^2}{2} - \frac{N^2}{2}\right]$$
$$= \infty - \infty = \text{ does not exist.}$$

In fact, if for instance we let $M = \sqrt{N^2 + 2A}$, where A is any real number such that $N^2 + 2A \geq 0$, then

$$\lim_{\substack{M\to\infty \\ N\to -\infty}} \left[\frac{M^2}{2} - \frac{N^2}{2}\right] = \lim_{N\to-\infty} \frac{2A}{2} = A$$

So, this double limiting process may produce any real number as limit. Similarly, we can make this double limit equal to $-\infty$ or ∞ or make it oscillate.

Definition 37. Suppose $y = f(x)$ real function continuous defined in the set $[a, c) \cup (c, b] \subset \mathbb{R}$ with a, b finite and at $x = c$, $y = f(x)$ is unbounded, that is, it approaches $\pm\infty$ as x approaches c. Then we define:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

where the two partial integrals have been defined in Definitions 34 and 35.

Instead of $[a, c) \cup (c, b]$ we could have $(a, c) \cup (c, b) \subset \mathbb{R}$ with a, b finite or infinite. Then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

in where the two partial integrals have been defined in Definitions 34, 35 and 36.

Example 5.1.6.

1.

$$\begin{aligned}
\int_{-2}^{3} \frac{dx}{(x-1)^3} &= \int_{-2}^{1} \frac{dx}{(x-1)^3} + \int_{1}^{3} \frac{dx}{(x-1)^3} \\
&= \lim_{\rho \to 1^-} \int_{-2}^{\rho} \frac{dx}{(x-1)^3} + \lim_{\sigma \to 1^+} \int_{\sigma}^{3} \frac{dx}{(x-1)^3} \\
&= \lim_{\rho \to 1^-} \left[-\frac{(x-1)^{-2}}{2} \right]_{-2}^{\rho} + \lim_{\sigma \to 1^+} \left[-\frac{(x-1)^{-2}}{2} \right]_{\sigma}^{3} \\
&= \lim_{\rho \to 1^-} \left[\frac{-1}{2(\rho-1)^2} + \frac{1}{18} \right] + \lim_{\sigma \to 1^+} \left[\frac{-1}{8} + \frac{1}{2(\sigma-1)^2} \right].
\end{aligned}$$

By manipulating the two limiting processes, this double limit may assume any possible value, finite or infinite. It follows that this improper integral does not exist.

2. Now we examine the following integral which, we must notice, is improper at $x = 1$:

$$
\begin{aligned}
\int_{-2}^{3} \frac{dx}{x-1} &= \int_{-2}^{1} \frac{dx}{x-1} + \int_{1}^{3} \frac{dx}{x-1} \\
&= \lim_{\rho \to 1^-} \int_{-2}^{\rho} \frac{dx}{x-1} + \lim_{\sigma \to 1^+} \int_{\sigma}^{3} \frac{dx}{x-1} \\
&= \lim_{\rho \to 1^-} [\ln|x-1|]_{-2}^{\rho} + \lim_{\sigma \to 1^+} \left[\ln|x-1|_{\sigma}^{3}\right] \\
&= \lim_{\rho \to 1^-} [\ln|\rho - 1| - \ln(3)] + \lim_{\sigma \to 1^+} [\ln(2) - \ln|\sigma - 1|] \\
&= [\ln(0^+) - \ln(3)] + [\ln(2) - \ln(0^+)] \\
&= [-\infty - \ln(3)] + [\ln(2) - (-\infty)] \\
&= -\infty + \infty = \text{does not exist}
\end{aligned}
$$

We could also write

$$
\begin{aligned}
&\lim_{\rho \to 1^-} [\ln|\rho - 1| - \ln(3)] + \lim_{\sigma \to 1^+} [\ln(2) - \ln|\sigma - 1|] \\
&\qquad = \ln\left(\frac{2}{3}\right) + \lim_{\substack{\rho \to 1^- \\ \sigma \to 1^+}} \ln\left(\frac{|\rho - 1|}{|\sigma - 1|}\right).
\end{aligned}
$$

We can easily see that the double limit $\lim\limits_{\substack{\rho \to 1^- \\ \sigma \to 1^+}} \ln\left(\frac{|\rho - 1|}{|\sigma - 1|}\right)$ can assume any value as $\sigma \to 1^+$and $\rho \to 1^-$independently. Therefore, the improper integral $\displaystyle\int_{-2}^{3} \frac{dx}{x-1}$ does not exist.

Notice that the point $x = 1$, at which this integral is improper, is an interior point of the interval of integration $[-2, 3]$. So, if we inadvertently write

$$\int_{-2}^{3} \frac{dx}{x-1} = [\ln|x-1|]_{-2}^{3} = \ln(2) - \ln(3) = \ln\left(\frac{2}{3}\right),$$

then we find a wrong answer, and we have made a bad mistake! We must also notice that the antiderivative of $f(x) = \frac{1}{x-1}$, $F(x) = \ln|x-1|$, is not defined (or continuous) at $x = 1$, a point inside the interval of integration $[-2, 3]$.

Remark 5.1.1. As we have seen in the previous three examples, whenever the final evaluation of an improper integral takes final formal form $\infty - \infty$, then the improper integral does not exist. By manipulating the limiting processes, we may make it assume any possible value, finite or infinite, and so such an improper integral does not exist. (Obviously $\infty + \infty = \infty$ and $-\infty - \infty = -\infty$.)

This should not be confused with the limits of the indeterminate form $\infty - \infty$. These limits may exist and are resolved by some mathematical manipulation and/or adjusting the well known $\mathbf{L'Hospital'srule}$. Also, as we will see in numerous cases and examples, when breaking an integral as a sum two integrals one of which is ∞ and the other one is $-\infty$, then this breaking is illegitimate and must be avoided. Otherwise, all sorts of mistakes can occur.

Example 5.1.7. 1. Similarly with the previous example,

$$\text{for } \infty \le a < 0 < b \le \infty, \text{ the integral } \int_{a}^{b} \frac{dx}{x} \text{ does not exist.}$$

Sometimes an integral seems to be improper, whereas it is proper. For instance, let us investigate the following two examples.

2. The integral

$$\int_{-1}^{1} \frac{\sin(x)}{x} dx$$

is proper, even though the function $f(x) = \dfrac{\sin(x)}{x}$ at $x = 0$ takes the indeterminate form $\frac{0}{0}$. This is so because, as we know from calculus,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Therefore, $f(x) = \dfrac{\sin(x)}{x}$ is bounded on the interval $[-1, 1]$ and can be continuously defined at $x = 0$, by assigning the value $f(0) = 1$.

This integral can be evaluated, by means of power series, as a series of real numbers. By using the power series expansion of the function $\sin(x)$ we find that the power series of the function $f(x)$ is:

$$f(x) = \frac{\sin(x)}{x} = \frac{\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!},$$

$\forall x \in \mathbb{R}$. Since we can integrate a power series, within its interval of convergence, term by term, we get

$$\begin{aligned}
\int_{-1}^{1} \frac{\sin(x)}{x} dx &= \int_{-1}^{1} \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} \right] dx \\
&= \sum_{n=0}^{\infty} \int_{-1}^{1} \left[(-1)^n \frac{x^{2n}}{(2n+1)!} \right] dx \\
&= \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!} \right]_{-1}^{1}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2}{(2n+1)(2n+1)!}$$

Remark 5.1.2. In the previous example, we have used the fact that we can integrate power series term by term, which means that we can commute the integral $\int_a^b$ with the infinite summation $\sum_{n=0}^{\infty} = \lim_{0 \le k \to \infty} \left(\sum_{n=0}^{k} \right)$. (That is, we can switch the order of integration and the limit process.) Whereas this is always legitimate with integrals of power series when the limits of integration a and b are inside their intervals of convergence, it does not hold in every situation with limits of sequences or series of functions, even if the limits of integration are within the domain of definition of all functions involved. Serious mistakes may occur if such a commutation is performed while it is not valid!

Example 5.1.8. As in the previous example so the following integral

$$\int_{-3}^{5} \frac{1-\cos(x)}{x^2} dx$$

is proper.

Again, at the singular point $x = 0$, the function $g(x) = \frac{1-\cos(x)}{x^2}$ is bounded and can be continuously defined by assigning the value $g(0) = \frac{1}{2}$. This follows from the fact that under certain hypotheses we can resolve a limit of type $\frac{0}{0}$ by using L ' Hôpital's rule. Indeed, we have:

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{1-\cos(x)}{x^2} = \lim_{x \to 0} \frac{[1-\cos(x)]'}{(x^2)'} = \lim_{x \to 0} \frac{\sin(x)}{2x} = \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

Remark 5.1.3. Under certain necessary conditions, we can resolve limits of the types $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ by using L' Hôpital's rule. These limits may assume any real value, or $\pm\infty$, or may not exist. If this problem arises

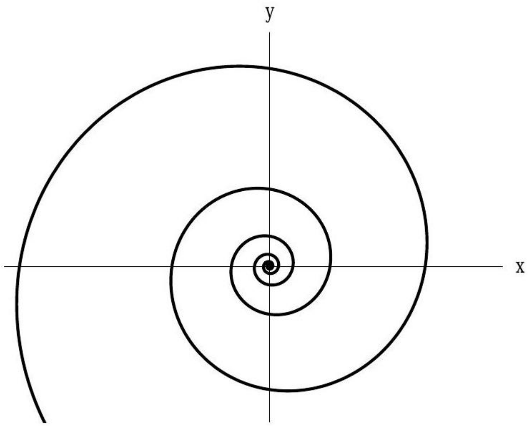

Figure 5.2: Logarithmic spiral $r = ae^{b\theta}$

at a point of a set over which we examine an integral and such a limit is equal to a real number, then the integral is proper with respect to this singular point. Otherwise, it is improper.

Applications

Application 1 : In calculus, geometry, differential geometry and other areas we encounter the **logarithmic spirals**. In polar coordinates (r, θ) they are given by the formula

$$r = ae^{b\theta},$$

where $a \neq 0$ and $b \neq 0$ real constants.

For such a curve $r = f(\theta)$ and $\theta_1 \leq \theta \leq \theta_2$, as we learn in calculus, the arc-length is given by

$$L(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Applying this to the logarithmic spiral, we find

$$L(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} |a|\sqrt{1+b^2}e^{b\theta} d\theta = |a|\frac{\sqrt{1+b^2}}{b}\left(e^{b\theta_2} - e^{b\theta_1}\right)$$

Now for $b > 0$ and any $\theta \in \mathbb{R}$ the $L(-\infty, \theta)$ is an improper integral but has finite value. Namely

$$L(-\infty, \theta) = |a|\frac{\sqrt{1+b^2}}{b}e^{b\theta}$$

Similarly for $b < 0$ and $\theta \in \mathbb{R}$, we get

$$L(\theta, \infty) = |a|\frac{\sqrt{1+b^2}}{-b}e^{b\theta}$$

Application 2 : In physics we learn that the Earth creates around it a conservative gravitational field. If the mass of the Earth is M, then the force W (weight) exerted on a mass m located at distance r from the center of gravity of the Earth, by Newton's law of gravitational attraction, has measure

$$W = -G\frac{Mm}{r^2}$$

where G is the universal gravitational constant. The minus sign has the meaning that the force is directed toward the center of gravity of the Earth.

The gravitational potential energy of m at a point P located at distance R from the center of gravity of the Earth O is equal to the work needed to move m from distance R to infinite distance. Then,

$$E = \int_R^\infty W(r)dr$$

Since the gravitational field is conservative (i.e., this integral is independent of the path), we can evaluate E by moving on the straight line OP from R to ∞, where O is considered to be the origin. So,

$$E = \int_R^\infty W(r)dr = \int_R^\infty -G\frac{Mm}{r^2}dr = -GMm\left[-\frac{1}{r}\right]_R^\infty = \frac{-GM}{R}m$$

The potential of the gravitational field of the Earth at any point P at distance R from O is defined to be the above energy E per unit-mass, and so it is

$$U = \frac{E}{m} = -\frac{GM}{R}$$

Application 3 : The decaying law of a radioactive substance is

$$m(t) = m_0 e^{kt}$$

where t is time, $m(t)$ is the radioactive mass remaining after time t, $m_0 = m(0)$ is the initial mass at time $t = 0$ and k is a negative constant representing the percentage rate of decay of the substance.

The mean life of an atom of this substance is

$$\mu = -k\int_0^\infty te^{kt}dt$$

We can compute the improper integral and find that the mean life, in fact, is

$$\begin{aligned}\mu &= -k\int_0^{\infty} te^{kt}dt \\ &= -k\int_0^{\infty} td\left(\frac{e^{kt}}{k}\right) \\ &= -k\left[t\frac{e^{kt}}{k}\right]_0^{\infty} + k\int_0^{\infty}\frac{e^{kt}}{k}dt \\ &= -[0-0] + k\left[\frac{e^{kt}}{k^2}\right]_0^{\infty} = -\frac{1}{k}\end{aligned}$$

[Notice that if $k < 0$, $\lim_{t\to\infty}\left(te^{kt}\right) = 0$ and $\lim_{t\to\infty}\left(e^{kt}\right) = 0.$]

Application 4 : The plane curve given implicitly by

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

has four cusps at the points $\{(1,0),(0,1),(-1,0),(0,-1)\}$. It is symmetrical about either axis, the lines $y = \pm x$ and about the origin. (See Figure 5.3.)

Then, by its symmetries, its arc-length (by the well-known formula from calculus) is going to be

$$L = 4\int_0^1 \sqrt{1+(y')^2}dx$$

By implicit differentiation, we find $\dfrac{dy}{dx} = -\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$ and so

$$\sqrt{1+(y')^2} = \sqrt{\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}} = \sqrt{\frac{1}{x^{\frac{2}{3}}}} = \frac{1}{x^{\frac{1}{3}}} = x^{\frac{-1}{3}}$$

So, even if the arc-length of this curve is finite, it is given by an improper

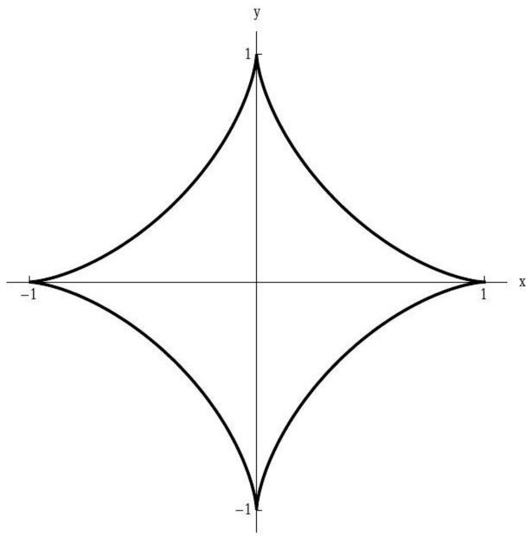

Figure 5.3: Astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$

integral as

$$L = 4\int_0^1 \sqrt{1 + (y')^2}dx = 4\int_0^1 x^{\frac{-1}{3}}dx = 4\left[\frac{x^{\frac{2}{3}}}{\frac{2}{3}}\right]_0^1 = 4 \cdot \frac{3}{2} = 6$$

Application 5 : If a company expects annual profits $p(t), t$ years from now with interest compounded continuously at an annual interest rate r, then the present value for all future profits, also called present value of the income stream $p(t)$, using appropriate Riemann sums, can be shown to be given by the improper integral

$$\begin{aligned}
\text{Present Value } &= \int_0^\infty e^{-rt}p(t)dt = \int_0^\infty p(t)d\left(\frac{e^{-rt}}{-r}\right) \\
&= \left[p(t)\frac{e^{-rt}}{-r}\right]_0^\infty + \int_0^\infty \frac{e^{-rt}}{r}dp(t) \\
&= \frac{p(0)}{r} + \frac{1}{r}\int_0^\infty e^{-rt}dp(t)
\end{aligned}$$

[We have assumed that $p(\infty) = \frac{e^{-r\infty}}{-r} = 0$, which is a natural condition.]

If we need to find the present value for a time interval $0 \leq a \leq b$ then we compute the above integral from a to b.

EXERCISES

1. Study the graphs of the functions $f(x), F(x), \bar{F}(x)$ and $F_c(x)$ of Example 5.1.2 by using the graphs and the information already provided and the information obtained by studying their first and second derivatives. Compare them with each other and observe the similarities and differences!

2. Prove that the function $F_c(x)$ in Example 5.1.2, is:

   (a) Continuous at every $x \in \mathbb{R}$.

   (b) Differentiable at every $x \in \mathbb{R}$, by showing:

   At the points $x = (2k+1)\pi$ with $k \in \mathbb{Z}$, use the definition of side derivatives and the help of L' Hôpital's rule to resolve the corresponding limits to show $F_c'[(2k+1)\pi] = f[(2k+1)\pi] = \frac{1}{3}$

Then show that at every $x \in \mathbb{R} F_c'(x) = f(x)$, where $f(x)$ is the given function in this example.

3. Check that for any real constant C

$$\frac{d}{dx}\left[\frac{1}{3}\arctan\left(\frac{3x\left(1-x^2\right)}{x^4-4x^2+1}\right)+C\right] = \frac{x^4+1}{x^6+1} > 0.$$

Using this, we find

$$\int_0^1 \frac{x^4+1}{x^6+1}dx = \left[\frac{1}{3}\arctan\left(\frac{3x\left(1-x^2\right)}{x^4-4x^2+1}\right)\right]_0^1 = 0-0=0$$

An integral of a positive function is 0(?)! Explain what has happened.

4. Someone wrote $\displaystyle\int_{-2}^{2} \frac{1}{x}dx = [\ln(|x|)]_{-2}^{2} = \ln(2)-\ln(2) = 0$, which is wrong. Find the error and explain why. (See Example 5.1.4 (1).)

5. Let

$$f(x) = \begin{cases} 1, & \text{if} \quad 0 \le x \le 1 \\ 2, & \text{if} \quad 1 < x \le 2 \end{cases}$$

Then $\int_0^2 f(x)dx = 3$ and an antiderivative of $f(x)$ is

$$F(x) = \begin{cases} x, & \text{if} \quad 0 \le x \le 1 \\ 2x, & \text{if} \quad 1 < x \le 2 \end{cases}$$

But, $F(2) - F(0) = 4 - 0 = 4 \neq 3$. Why has this happened?

Find an antiderivative $G(x)$ of $f(x)$ such that $G(2) - G(0) = 3$, the correct answer.

6. **Project on periodic real functions**.

In this project, without loss of generality, we consider real functions of a real variable $y = f(x) : \mathbb{R} \longrightarrow \mathbb{R}$. Such a function is called **periodic** if there is a real number $q \neq 0$ such that $f(x) = f(x + q), \quad \forall x \in \mathbb{R}$. This number q is called a **period of the real function** $y = f(x)$. Otherwise, $y = f(x)$ is called **non − periodic**.

Obviously, $q = 0$ satisfies this condition for every function. So, $q = 0$ does not tell us anything about any function. We can call $q = 0$ **trivial period** for any function.

Also, if $y = f(x)$ is a constant function, then obviously any real number $q \in \mathbb{R}$ is a period of it.

[In integrals that we study in this text, at times, we use properties (g) and (h) below. With this opportunity, we try to present a more complete exposition of the periodic real functions.]

(a) If functions $y = f(x)$ and $y = g(x)$ have a common period q and $c \in \mathbb{R}$ is a constant. then prove that the functions $f + q, f - q,\ c \cdot f,\ f \cdot g, \frac{f}{g}$ and $f \circ g$ have q as a period. For the composition $f \circ g$ we can relax one hypothesis. Which one and why?

(b) For any q period of $y = f(x)$, prove that $-q$ and in general any kq with $k \in \mathbb{Z}$ (integer) is another period.
If moreover r is any other period of $y = f(x)$ (including the trivial one), then $q \pm r$ is also a period, but qr and $\frac{q}{r}$ may not be periods of $y = f(x)$

(c) For any real numbers $s \neq t$, we define a so-called Dirichlet function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$y = f(x) = \begin{cases} s, & \text{if } x = \text{ rational} \\ t, & \text{if } x = \text{ irrational} \end{cases}$$

In the literature, many times, we encounter such a function with $s = 0$ and $t = 1$ which is the characteristic function of the irrationals in $\mathbb{R}, \chi_{\mathbb{R}-\mathbb{Q}}$, or with $s = 1$ and $t = 0$ which is the characteristic function of the rationals in $\mathbb{R}, \chi_{\mathbb{Q}}$.

For any Dirichlet function $y = f(x)$, prove:

i. It is nowhere continuous.

ii. It is even [i.e., $f(-x) = f(x)$].

iii. It is periodic and any rational number $r \in \mathbb{Q}$ is a period.

iv. Any irrational number $w \in \mathbb{R} - \mathbb{Q}$ is not a period of $y = f(x)$.

(d) If $y = f(x)$ possesses a point of continuity (i.e., there is an $x_0 \in \mathbb{R}$ such that $\lim_{x \to x_0} f(x) = f(x_0) \left[= f\left(\lim_{x \to x_0} x \right) \right]$) and a sequence of nonzero periods ($q_n \neq 0$) with $n \in \mathbb{N}$ such that $\lim_{n \to \infty} q_n = 0$, then $y = f(x)$ is identically constant.

(e) If $y = f(x)$ is periodic, non-constant and possesses a point of continuity, then it cannot have a sequence of distinct periods that converges to zero.

Then prove that such a function has a minimum positive period and any other period is an integer multiple of it.

i.e., if $p := \inf\{$ positive periods of $y = f(x)\}$, then $p > 0$ and

$f(x) = f(x+p), \quad \forall x \in \mathbb{R}$. Hence,

$$\begin{aligned} p &:= \inf\{\text{ positive periods of } y = f(x)\} \\ &= \min\{\text{ positive periods of } y = f(x)\} \end{aligned}$$

Moreover, for any other period q of $y = f(x)$, there is $k \in \mathbb{Z}$ such that $q = kp$.

In such a case, this number p is unique and we call it the period of the real function $y = f(x)$. Then the function $y = f(x)$ is called p-periodic

(f) Give an example of two $p-$ periodic functions $y = f(x)$ and $y = g(x)$ such that the period q of their sum $f+g$, or difference $f-g$, or product fg and/or ratio $\frac{f}{g}$ is not p. What are the possible answers that the ratio $\frac{q}{p}$ may assume?

(g) If a periodic function $y = f(x)$ is Riemann integrable, then for any $a \in \mathbb{R}$ and any of its periods $q \in \mathbb{R}$ the integral

$$\int_a^{a+q} f(x)dx$$

is fixed, that is, independent of $a \in \mathbb{R}$. (For $q = 0$ this is trivially true regardless.)

(h) Suppose $f : [0, \infty) \longrightarrow \mathbb{R}$ is a periodic function with period $p > 0$ which is Riemann integrable in every interval $[0, M]$, with $M > 0$ Then for every $u \geq 0$ prove

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(t)dt = \frac{1}{p} \int_0^p f(x)dx = \frac{1}{p} \int_u^{u+p} f(x)dx$$

[Hint: For any $x > 0$, consider $n := \left[\left[\frac{x}{p}\right]\right]$ the integer part of

$\frac{x}{p}$. Then use:

$$\frac{1}{x}\int_0^x f(t)dt = \frac{1}{x}\left[\sum_{k=0}^{n-1}\int_{kp}^{(k+1)p} f(t)dt + \int_{np}^{x} f(t)dt\right]$$

the previous result, the inequality $\frac{1}{p} > \frac{n}{x} > \frac{n}{(n+1)p}$ (prove it first), the Squeeze Lemma, etc.]

(i) If $y = f(x)$ is periodic and differentiable, then its derivative $y' = f'(x)$ is also periodic with the same periods as $y = f(x)$ and it is zero at at least one point in every interval of length greater than or equal to any positive period.

(j) Give an example of a periodic function $y = f(x)$ whose integral

$$F(x) = \int_0^x f(t)dt$$

is not periodic.

(k) Give examples of periodic functions $y = f(x)$ with two irrational periods a and b, such that $\frac{a}{b}$ is rational.

(l) Give an example of two non-periodic functions whose composition is periodic.

(m) Can the composition of a periodic function and a non-periodic function in either order be periodic?

(n) If $y = f(x)$ is a continuous and periodic function with an irrational period q, then prove that the set $f(\mathbb{Z}) := \{f(n) \mid n \in \mathbb{Z}\}$ is dense in its range $f(\mathbb{R}) := \{f(x) \mid x \in \mathbb{R}\})$. That is, between any two different numbers in the range $f(\mathbb{R})$ there is a number of the set $f(\mathbb{Z})$. This is equivalent to the fact that any number in the range $f(\mathbb{R})$ is the limit of a sequence in $f(\mathbb{Z})$. (The latter statement is easier to prove.)

(o) Using the previous result and the properties of the trigonometric functions $y = \cos(x)$ and $y = \sin(x)$, prove that the sets $\{\cos(n) \mid n \in \mathbb{N}\}$ and $\{\sin(n) \mid n \in \mathbb{N}\}$ are dense in the range $[-1, 1]$

(p) If $y = f(x)$ is periodic and has two periods a and b such that $\frac{a}{b}$ is irrational, then it has a sequence of different periods that converges to zero. In such a case, in order for $y = f(x)$ to be non-constant by (4.) above] it must be discontinuous everywhere.

(q) Give (construct) an example of a non-constant function with two periods a and b such that $\frac{a}{b}$ is irrational.

[Hint: In items (k), (m) and (n) you can use the following fact: "For any irrational number t the set $\mathbb{Z} + t\mathbb{Z}$ is dense in $\mathbb{R}$. I.e., between any two different real numbers there is a number of the form $k + tl$, with k and l integers." You may provide a proof of this fact, but if you cannot, just use it readily.]

7. **Project on the modified Dirichlet function**.

Part I :

For any rational number $r \in \mathbb{Q}$ we consider two integers $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$ such that: $q > 0$, p and q have no common factors except the trivial 1 [i.e., $\gcd(p, q) = 1$] and $r = \frac{p}{q}$. For any integer m (including $m = 0$) we have $m = \frac{m}{1}$ and so $p = m$ and $q = 1$. We call such a representation of the rational number r **reduced representation**.

With this in mind we define the so-called **modified Dirichlet** or

Riemann function $g : \mathbb{R} \longrightarrow \mathbb{R}$ by:

$$y = g(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ rational in reduced representation} \\ 0, & \text{if } x = \text{ irrational.} \end{cases}$$

Now prove:

(a) $\forall x \in \mathbb{R}, 0 \leq g(x) \leq 1, \quad g^{-1}(\{0\}) = \mathbb{R} - \mathbb{Q} \quad$ and $g^{-1}(\{1\}) = \mathbb{Z}$

(b) $\forall w \in \mathbb{R}, \lim\limits_{\substack{x \to w \\ x \neq w}} g(x) = 0.$

(c) $y = g(x)$ is even [i.e., $g(-x) = g(x)$].

(d) $y = g(x)$ is periodic and the set of its periods is exactly $\mathbb{Z}$.

(e) $y = g(x)$ is nowhere differentiable.

Part II :

We consider the function $y = h(x)$ to be the restriction of $y = g(x)$ on the closed interval $[0, 1]$. (In general we could consider any interval $[a, b]$ where $-\infty < a < b < \infty$, but we use $[0, 1]$ without loss of generality.) I.e., $h : [0, 1] \longrightarrow \mathbb{R}$ is defined by:

$$y = h(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ rational in reduced representation in } [0,1] \\ 0, & \text{if } x = \text{ irrational in } [0,1]. \end{cases}$$

Then:

(a) Prove that $y = h(x)$ is Riemann integrable and $\int_0^1 h(x)dx = 0$. (Read again the definition of "Riemann integrable function" and/or some criteria of "Riemann integrability" from appropriate books of Mathematical Analysis and apply them to this function.)

(b) Define $u : [0,1] \longrightarrow \mathbb{R}$ by

$$y = u(x) = \begin{cases} 1, & \text{if } 0 < x \leq 1 \\ 0, & \text{if } x = 0 \end{cases}$$

Prove that $y = u(x)$ is Riemann integrable and $\int_0^1 u(x)dx = 1$

(c) Prove that

$$y = (u \circ h)(x) = \chi_{[0,1]\cap\mathbb{Q}}(x) = \begin{cases} 1, & \text{if } x = \text{ rational in } [0,1] \\ 0, & \text{if } x = \text{ irrational in } [0,1] \end{cases}$$

Then prove that the composition of these two Riemann integrable functions, $u \circ h = \chi[0,1] \cap \mathbb{Q}$, is not a Riemann integrable function.

(d) However, $u \circ h = \chi_{[0,1]\cap Q}$ is the point-wise limit of a sequence of Riemann integrable functions, defined as follows:

Let $[0,1] \cap \mathbb{Q} = \{r_1, r_2, r_3, \ldots,\}$ be an enumeration of the rational numbers in $[0,1]$. Then for all $n \in \mathbb{N}$ define

$$y = v_n(x) = \begin{cases} 1, & \text{if } x \in \{r_1, r_2, \ldots, r_n\} \\ 0, & \text{if } x \in [0,1] - \{r_1, r_2, \ldots, r_n\} \end{cases}$$

Now prove that: $\forall n \in \mathbb{N}, y = v_n(x)$ is a Riemann integrable function with $\int_0^1 v_n(x)dx = 1$ and $\forall x \in [0,1], \lim\limits_{n\to\infty} v_n(x) = \chi_{[0,1]\cap\mathbb{Q}}(x)$

(e) Finally, prove that: $\lim\limits_{m\to\infty}\left[\lim\limits_{n\to\infty}\cos^{2n}(m!\pi x)\right] = \chi_{[0,1]\cap\mathbb{Q}}(x)$
That is, $\chi_{[0,1]\cap\mathbb{Q}}(x)$ is an iterated limit of a double limit process of bounded continuous functions.

5.2 Cauchy Principal Value

In some cases we can define the so-called **Cauchy principal value** or simply **principal value** of an improper integral. This is a certain symmetrical limit and it is defined in the following four situations:

Definition 38. If the integral is improper simply because the set of integration is $\mathbb{R} = (-\infty, \infty)$, then we define its principal value to be:

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx \stackrel{\text{def}}{=} \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$$

Definition 39. If the set of integration is $[a, c) \cup (c, b]$, where $a < c < b$ finite real numbers, and the integral becomes improper at c, then we define its principal value to be:

$$\text{P.V.} \int_{a}^{b} f(x)dx \stackrel{\text{def}}{=} \lim_{\epsilon\to 0+} \left[\int_{a}^{c-\epsilon} f(x)dx + \int_{c+\epsilon}^{b} f(x)dx \right]$$

Definition 40. If both situations of the previous two definitions occur, i.e., we have improper integrals over $(-\infty, c) \cup (c, \infty)$, with $c \in \mathbb{R}$, then we combine the two definitions and we define the principal value of this improper integral to be:

$$P.V. \int_{\infty}^{\infty} f(x)dx \stackrel{\text{def}}{=} \lim_{\substack{\epsilon\to 0+ \\ R\to\infty}} \left[\int_{-R}^{c-\epsilon} f(x)dx + \int_{c+\epsilon}^{R} f(x)dx \right]$$

Definition 41. If the set of integration is the finite open interval (a, b) ($a < b$ are finite real numbers), and the integral is improper just because the interval is open at both endpoints, then we define the principal value of this improper integral to be:

$$P.V. \int_a^b f(x)dx \stackrel{\text{def}}{=} \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b-\epsilon} f(x)dx$$

Again we see that the principal values are obtained by symmetrical limiting processes and therefore are special. However, they turn out to be very useful in mathematics and applications. We will see applications of the principal value in many sections that follow.

Example 5.2.1. 1.

$$\begin{aligned}
\text{P.V.} \int_{-\infty}^{\infty} xdx &= \lim_{R\to\infty} \int_{-R}^{R} xdx \\
&= \lim_{R\to\infty} \left[\frac{x^2}{2}\right]_{-R}^{R} \\
&= \lim_{R\to\infty} \left[\frac{R^2}{2} - \frac{(-R)^2}{2}\right] \\
&= \lim_{R\to\infty} 0 = 0
\end{aligned}$$

2.

$$\begin{aligned}
\text{P.V.} \int_{-\infty}^{\infty} x^2dx &= \lim_{R\to\infty} \int_{-R}^{R} x^2dx \\
&= \lim_{R\to\infty} \left[\frac{x^3}{3}\right]_{-R}^{R} \\
&= \lim_{R\to\infty} \left[\frac{R^3}{3} - \frac{(-R)^3}{3}\right] \\
&= \lim_{R\to\infty} \frac{2R^3}{3} = \infty
\end{aligned}$$

3. As we have seen in Example 5.1.7 (1) the integral $\int_{-1}^{1} \frac{dx}{x}$ does not exist, but

$$\begin{aligned}
\text{P.V.} \int_{-1}^{1} \frac{dx}{x} &= \lim_{\epsilon \to 0^+} \left[\int_{-1}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{1} \frac{dx}{x} \right] \\
&= \lim_{\epsilon \to 0^+} \left([\ln |x|]_{-1}^{-\epsilon} + [\ln |x|]_{\epsilon}^{1} \right) \\
&= \lim_{\epsilon \to 0^+} (\ln \epsilon - \ln 1 + \ln 1 - \ln \epsilon) \\
&= \lim_{\epsilon \to 0^+} 0 = 0
\end{aligned}$$

4. Similarly $\int_{-2}^{3} \frac{dx}{x}$ does not exist, but

$$\begin{aligned}
P.V. \int_{-2}^{3} \frac{dx}{x} &= \lim_{\epsilon \to 0^+} \left[\int_{-2}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{3} \frac{dx}{x} \right] \\
&= \lim_{\epsilon \to 0^+} \left([\ln |x|]_{-2}^{-\epsilon} + [\ln |x|]_{\epsilon}^{3} \right) \\
&= \lim_{\epsilon \to 0^+} (\ln \epsilon - \ln 2 + \ln 3 - \ln \epsilon) \\
&= \lim_{\epsilon \to 0^+} \ln \left(\frac{3}{2} \right) \\
&= \ln \left(\frac{3}{2} \right).
\end{aligned}$$

5. Also $\int_{-\infty}^{\infty} \frac{dx}{x}$ does not exist, but

$$\begin{aligned}
\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x} &= \lim_{\substack{\epsilon\to 0^+ \\ R\to\infty}} \left[\int_{-R}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{R} \frac{dx}{x}\right] \\
&= \lim_{\substack{\epsilon\to 0^+ \\ R\to\infty}} \left([\ln|x|]_{-R}^{-\epsilon} + [\ln|x|]_{\epsilon}^{R}\right) \\
&= \lim_{\substack{\epsilon\to 0^+ \\ R\to\infty}} (\ln\epsilon - \ln R + \ln R - \ln\epsilon) \\
&= \lim_{\substack{\epsilon\to 0^+ \\ R\to\infty}} 0 = 0.
\end{aligned}$$

6.

$$\begin{aligned}
\text{P.V.} \int_{-2}^{3} \frac{dx}{(x-1)^3} &= \lim_{\epsilon\to 0^+} \left[\int_{-2}^{1-\epsilon} \frac{dx}{(x-1)^3} + \int_{1+\epsilon}^{3} \frac{dx}{(x-1)^3}\right] \\
&= \lim_{\epsilon\to 0^+} \left(\left[\frac{-1}{2(x-1)^2}\right]_{-2}^{1-\epsilon} + \left[\frac{-1}{2(x-1)^2}\right]_{1+\epsilon}^{3}\right) \\
&= \lim_{\epsilon\to 0^+} \left(\frac{-1}{2\epsilon^2} + \frac{1}{18} - \frac{1}{8} + \frac{1}{2\epsilon^2}\right) \\
&= \frac{1}{18} - \frac{1}{8} = \frac{-5}{72}
\end{aligned}$$

7.

$$\begin{aligned}
\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x^2+1} &= \lim_{R\to\infty} [\arctan(x)]_{-R}^{R} \\
&= \lim_{R\to\infty} [\arctan(R) - \arctan(-R)] \\
&= \lim_{R\to\infty} \arctan(R) - \lim_{R\to\infty} \arctan(-R) \\
&= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi
\end{aligned}$$

since both of the two partial limits exist.

Note 5.2.2. By the above definitions and examples we conclude the following:

1. If the improper integral exists, then all limiting processes give the same answer which is the value of the improper integral, and so the principal value also exists and it is equal to the improper integral.

2. If the improper integral does not exist, then its principal value may or may not exist.

3. If the principal value does not exist, then the improper integral does not exist either, since the principal value is one of the limiting processes.

Thus, the principal value of an improper integral constitutes a proper generalization of the improper integral. When we know á-priori that the improper integral exists, we can evaluate it by just computing its principal value, especially when the computation of this symmetric limit is easier than any other way.

The following definitions and immediate results are also useful:

(a) If $y = f(x)$ is an odd function in $\mathbb{R}$, i.e., by definition

$\forall x \in \mathbb{R}, f(-x) = -f(x)$, then

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = 0$$

(b) If $y = f(x)$ is an even function in $\mathbb{R}$, i.e., by definition $\forall x \in \mathbb{R}, f(-x) =$

$f(x)$, then

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx =$$
$$2 \lim_{R\to\infty} \int_{0}^{R} f(x)dx = 2 \lim_{R\to\infty} \int_{-R}^{0} f(x)dx$$

In general, a function $y = f(x)$, where $f : \mathbb{R} \longrightarrow \mathbb{R}$, is odd about a point $c \in \mathbb{R}$, if by definition

$$\forall u \in \mathbb{R}, \quad f(c-u) = -f(c+u), \text{ or } \forall x \in \mathbb{R}, \quad f(2c-x) = -f(x)$$

Also, $y = f(x)$ is even about a point $c \in \mathbb{R}$, if by definition

$$\forall u \in \mathbb{R}, \quad f(c-u) = f(c+u), \text{ or } \forall x \in \mathbb{R}, \quad f(2c-x) = f(x)$$

Now consider $y = f(x)$ a function defined in $(-\infty, c) \cup (c, \infty)$, with $c \in \mathbb{R}$. We have:

(c) If $y = f(x)$ is an odd function about c [we let $f(c) = 0$, in this case], then

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{\substack{\epsilon\to 0+\\ R\to\infty}} \int_{-R}^{c-\epsilon} f(x)dx + \lim_{\substack{\epsilon\to 0+\\ R\to\infty}} \int_{c+\epsilon}^{R} f(x)dx = 0$$

(d) If $y = f(x)$ is an even function about c, then

$$\begin{aligned}
\text{P.V.} \int_{-\infty}^{\infty} f(x)dx &= \lim_{\substack{\epsilon\to 0^+\\ R\to\infty}} \int_{-R}^{c-\epsilon} f(x)dx + \lim_{\substack{\epsilon\to 0^+\\ R\to\infty}} \int_{c+\epsilon}^{R} f(x)dx \\
&= 2 \lim_{\substack{\epsilon\to 0^+\\ R\to\infty}} \int_{c+\epsilon}^{R} f(x)dx \\
&= 2 \lim_{\substack{\epsilon\to 0^+\\ R\to\infty}} \int_{-R}^{c-\epsilon} f(x)dx..
\end{aligned}$$

(e) Rule of translate or shift: Consider a function

$$f : [a, a+r] \longrightarrow \mathbb{R}, \qquad \text{with} \qquad r > 0$$

The translate or shift of $y = f(x)$ at the interval $[b, b+r]$ is given by

$$y = f(x - b + a), \qquad \text{with} \qquad b \le x \le b + r$$

EXERCISES

1. Give all the reasons as to why the following integrals are improper:

$$\begin{aligned}
I_1 &= \int_0^{\infty} \ln(x)dx, & I_2 &= \int_{-\infty}^{\infty} \frac{\sin(x)}{x}dx \\
I_3 &= \int_0^{\infty} \frac{\cos(x)}{x}dx, & I_4 &= \int_0^{\infty} \frac{1}{2x-1}dx \\
I_5 &= \int_0^{\infty} \frac{\sin(5x)}{e^{2x}-1}dx, & I_6 &= \int_0^{\infty} x^n e^{-x}dx, \quad n \in \mathbb{Z}
\end{aligned}$$

In problems 2-22 compute the given improper integrals. (Prove that they are equal to $-\infty$, or $+\infty$, or the provided value and/or you find their values. In some of these problems you have to distinguish different cases depending on the values of the parameters involved.)

2. $\int_0^\infty e^{-\mu x} dx,$ where $\mu \in \mathbb{R}$ constant.
3. $\int_1^\infty x^\alpha \ln(x) dx$ and $\int_0^1 x^\alpha \ln(x) dx$ where α is a real constant
4. $\int_{-\infty}^\infty \frac{|x|}{x^2+1} dx$ and P.V. $\int_{-\infty}^\infty \frac{|x|}{x^2+1} dx$
5. $\int_0^9 \frac{x}{(x-3)^2} dx$ and P.V. $\int_0^9 \frac{x}{(x-3)^2} dx$
6. $\int_a^\infty \frac{dx}{x^p}$ where $a > 0$ and p are real constants.
7. $\int_2^\infty \frac{dt}{t[\ln(t)]p}$ where $p \geq 1$ is a real constant.
8. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan(x) dx$ and P.V. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan(x) dx$
9. $\int_0^\infty \frac{dx}{(x+1)^3}$ and $\int_0^\infty \frac{dx}{(x-1)^3}$
10. $\int_1^\infty \frac{dx}{(x-2)^3}$ and P.V. $\int_1^\infty \frac{dx}{(x-2)^3}$
11. $\int_{-\infty}^\infty |x| e^{-x^2} dx$
12. $\int_{-\infty}^\infty e^{-x} \cos(x) dx$ and $\int_{-\infty}^\infty e^{-x} \sin(x) dx$
13. $\int_3^\infty \frac{dx}{x^2+x-2} = \frac{1}{3} \ln\left(\frac{5}{2}\right)$
14. $\int_0^\infty e^{-\alpha x} \sin(\beta x) dx = \frac{\beta}{\alpha^2+\beta^2}$ where $\alpha > 0$ and $\beta \in \mathbb{R}$ constants.
15. $\int_a^\infty e^{-\alpha x} \sin(\beta x) dx$ where $\alpha > 0, \beta \in \mathbb{R}$ and $a \in \mathbb{R}$ constants.
16. $\int_0^\infty e^{-\alpha x} \cos(\beta x) dx = \frac{\alpha}{\alpha^2+\beta^2}$ where $\alpha > 0$ and $\beta \in \mathbb{R}$ constants.
17. $\int_a^\infty e^{-\alpha x} \cos(\beta x) dx$ where $\alpha > 0, \beta \in \mathbb{R}$ and $a \in \mathbb{R}$ constants.
18. $\int_0^1 x^{p-1} dx$ where p is a real constant.

19.
$$\text{(a)} \int_0^a \ln(x)dx \quad \text{and} \quad \int_a^\infty \ln(x)dx$$
$$\text{(b)} \int_0^a \ln^2(x)dx \quad \text{and} \quad \int_a^\infty \ln^2(x)dx$$

where $a > 0$ is a real constant.

20. $\int_{-\infty}^{\infty} \frac{dx}{x^2+2ax+b^2} = \frac{\pi}{\sqrt{b^2-a^2}}$ for $b > |a|$ real constants.

21. By making two successive appropriate u-substitutions, prove that

$$\int_3^\infty \frac{dx}{x\ln(x)[\ln[\ln(x)]]^2} = \frac{1}{\ln[\ln(3)]}$$

22. By making two successive appropriate u-substitutions, prove that

$$\int_3^\infty \frac{dx}{x\ln(x)\ln[\ln(x)]} = \infty$$

23. Show that the integral

$$\int_{-1}^{1} \frac{e^x - 1}{x} dx$$

is proper and find its value as a series of real numbers.

24. Compute the integrals

$$\int_{-2}^{2} \frac{1-\cos(x)}{x^2} dx \quad \text{and} \quad \int_{-3}^{5} \frac{1-\cos(x)}{x^2} dx$$

as series of real numbers.

[Hint: Use the power series of $\cos(x)$, etc.]

25. Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Prove:

(a) If $y = f(x)$ is odd, then $f(0) = 0$

(b) If $y = f(x)$ is both odd and even, then it is identically zero.

(c) If $y = f(x)$ is differentiable and odd, then $f'(x)$ is even.

(d) If $y = f(x)$ is differentiable and even, then $f'(x)$ is odd.

26. (a) Prove that with compositions of functions we have the rules:

$$\begin{aligned} \text{even o even} &= \text{even} \\ \text{even o odd} &= \text{even} \\ \text{odd o even} &= \text{even} \\ \text{odd o odd} &= \text{odd.} \end{aligned}$$

(b) Prove that with multiplications of functions we have the rules:

$$\begin{aligned} \text{even} \cdot \text{even} &= \text{even} \\ \text{even} \cdot \text{odd} &= \text{odd} \\ \text{odd} \cdot \text{even} &= \text{odd} \\ \text{odd} \cdot \text{odd} &= \text{even.} \end{aligned}$$

27. Consider a function $f : \mathbb{R} \to \mathbb{R}$. Then prove that any two of the following statements imply the third one: (a) $f(x)$ is odd about $x = 0$, i.e., $f(-x) = -f(x)$. (b) $f(x)$ is odd about $x = c$, i.e., $f(2c-x) = -f(x)$ or $f(c-u) = -f(c+u)$ (c) $f(x)$ is $2c-$ periodic, i.e., $f(x + 2c) = f(x)$

28. Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then prove that any two of the following statements imply the third one:

(a) $f(x)$ is even about $x = 0$, i.e., $f(-x) = f(x)$.

(b) $f(x)$ is even about $x = c$, i.e., $f(2c - x) = f(x)$ or $f(c - u) = f(c + u)$

(c) $f(x)$ is $2c$-periodic, i.e., $f(x + 2c) = f(x)$

29. Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then prove:

(a) If $f(x)$ is odd about $x = 0$ and even about $x = c$, then $f(x)$ is $4c-$ periodic, i.e., $f(x + 4c) = f(x)$.

(b) If $f(x)$ is even about $x = 0$ and odd about $x = c$, then $f(x)$ is $4c-$ periodic, i.e., $f(x + 4c) = f(x)$.

(c) Notice that here we cannot have results similar to the results of the two previous problems. Why?

30. Consider any $c > 0$ and

$$f(x) = x(c - x), \quad 0 \leq x \leq c$$

Show that the extension of this function all over $(-\infty, \infty)$. such that the extended function is odd about both $x = 0$ and $x = c$, is given by:

$\forall n \in \mathbb{Z}$, for $nl \leq x \leq (n + 1)c$, then $f(x) = (-1)^n(x - nc)[(n + 1)c - x]$.

31. For any $a \in$ R and $p > 0$, consider any real function $y = f(x)$ defined in the interval $[a, a + p)$ or $(a, a + p]$

(a) Extend this function to the whole R periodically with a period equal to p

(b) Give an example in which the period of the extended function in (a) is less than p.

(c) If we consider $f(x)$ defined on the closed interval $[a, a+p]$, then give an example of a function $f(x)$ which cannot be extended as a periodic function.

(d) Under what condition a real function $f(x)$ defined on the closed interval $[a, a+p]$ can be extended as a periodic function to the whole R and with a period equal to p?

5.3 Some Criteria of Existence

We have defined the improper integrals as certain limits. These limits may or may not exist. When such a limit exists, we say that the improper integral exists or it is convergent. If the limit does not exist, then we say that the improper integral does not exist or it is divergent.

In the previous definitions, for more generality, the real value function $y = f(x)$ was considered to be piecewise continuous rather than continuous. (In the most general theory of integration developed in Advanced Real Analysis, we deal with more general integrals of a "very large" class of functions, the class of the measurable functions. We study these in an advanced course of real analysis.)

Necessary and sufficient conditions for the existence of improper integrals are developed in advanced calculus, mathematical analysis and real analysis. So, we will content ourselves with the few criteria stated in this section, some of which are reminiscent to criteria for the convergence of infinite series in calculus. These criteria are sufficient and powerful enough to give answers about existence or non-existence (convergence or non-convergence) questions for almost all the interesting improper integrals of mathematics and scientific applications at this primary level.

Definition 42. Non – standard Definition : In this book, we shall call a function to be *a* "**nice function**" if it is piecewise continuous in its domain of definition with finitely many discontinuities, each of which is of the following three types:

1. Jump discontinuity with finite or infinite jump.
2. Essential discontinuity because the limit of the function, as x approaches the point of discontinuity, is $\pm\infty$.
3. Essential discontinuity, such that the limit of the function, as x approaches the point of discontinuity, does not exist and is not $\pm\infty$. (The function oscillates.) In this case, we shall assume the extra condition that the function is bounded in some interval containing the point of the essential discontinuity.

The continuous functions are of course a subset of this set of nice functions since they have zero discontinuities in their domain. We use this non-standard term of "nice functions" for short, so that we do not have to repeat these conditions whenever we need them throughout this book. So, from now on we must remember what we mean by this non-standard term of "nice function" whenever we refer to it.

When a jump discontinuity has infinity jump or the limit of the function at the point of the discontinuity is $\pm\infty$, then the function is unbounded. In general, the discontinuity is essential if the limit of the function at the point of the discontinuity does not exist.

Also the domain of definition of such a function is going to be denoted by a capital letter like A, where $A \subseteq \mathbb{R}$ is any set that we have already encountered in the definitions of the previous two sections and/or any nice set that we have already dealt with in an undergraduate calculus course (e.g., a bounded closed interval, a finite union of bounded closed intervals, etc.)

Theorem 5.3.1. *(***Comparison Test with Non−negative Functions***)* *Let f and g be two nice functions defined in a set $A \subset \mathbb{R}$ (as we have indicated in the previous paragraph) and satisfying the inequality $0 \leq f(x) \leq g(x)$. Then we have:*

(a) If $\int_A g(x)dx$ exists, then $\int_A f(x)dx$ exists.

["Exists" for a non-negative function $0 \leq f(x) \leq g(x)$ means that the integral assumes a finite non-negative value.]

In this case we have the inequality

$$0 \leq \int_A f(x)dx \leq \int_A g(x)dx < \infty$$

(b) If $\int_A f(x)dx$ does not exist, then $\int_A g(x)dx$ does not exist.

["Does not exist" for non-negative functions $0 \leq f(x) \leq g(x)$ means that the integrals are infinite, that is, their values are equal ∞.So, in this case we have $\int_A f(x)dx = \infty = \int_A g(x)dx$.]

Proof. The proof of this criterion is rather obvious, since for any closed interval $[p, q] \subseteq A$ and for any nice function satisfying the inequality $0 \leq f(x) \leq g(x)$, by basic calculus we have

$$0 \leq \int_p^q f(x)dx \leq \int_p^q g(x)dx$$

and the limiting processes preserve the $\leq$ inequalities. Then, to prove claim (a) and claim (b) of the Theorem, we respectively use the facts

$$\int_A g(x)dx < \infty \quad \text{and} \quad \int_A f(x)dx = \infty$$

(Note: This criterion is reminiscent of the Comparison Test for convergence of non-negative series.) □

Example 5.3.2.

1. Prove that
$$\int_0^\infty e^{-x^2} dx \text{ exists.}$$
(In other words, it is convergent or equals a finite value.)

Consider the continuous function $f(x) = e^{-x^2}$ on $[0, \infty)$ and define

$$g(x) = \begin{cases} f(x) & \text{for } 0 \le x \le 1 \\ e^{-x} & \text{for } 1 \le x < \infty \end{cases}$$

Then $0 < f(x) \le g(x), \forall x \in [0, \infty)$ and

$$\int_0^\infty g(x)dx = \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x} dx = \int_0^1 e^{-x^2} dx + \left[-e^{-x}\right]_1^\infty =$$
$$(\text{ finite value }) + \left[0 - \left(-e^{-1}\right)\right] = (\text{ finite value })$$

Therefore.

$$\int_0^\infty f(x)dx = \int_0^\infty e^{-x^2} dx < \int_0^\infty g(x)dx < \infty$$

and so $\int_0^\infty f(x)dx$ is finite.

In the above inequality we have used the strictly less because $f(x) < g(x), \quad \forall x \in (1, \infty)$

In the same way we can prove that

$$\int_{-\infty}^\infty e^{-x^2} dx \text{ exists}$$

or we can use the fact that $f(x) = e^{-x^2}$ with $x \in \mathbb{R}$ is an even function and so

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2 \int_{0}^{\infty} e^{-x^2} dx$$

Note : There is no explicit antiderivative of e^{-x^2}. This is proven by **Liouville′s theoryfor finding antiderivatives in terms of elementary functions**.

2. Prove that

$$\int_{2}^{\infty} \frac{1}{\ln(x)} dx = \infty$$

i.e., it diverges.

On $[2, \infty)$ we have that $0 < \frac{1}{x} < \frac{1}{\ln(x)}$.

Now

$$\int_{2}^{\infty} \frac{1}{x} dx = [\ln(x)]_2^{\infty} = \infty$$

Therefore,

$$\int_{2}^{\infty} \frac{1}{\ln(x)} dx = \infty$$

Again, there is no explicit antiderivative for $\int \frac{1}{\ln(x)} dx$

Now we state the Integral Test, already known from calculus for checking the convergence or divergence of certain infinite series. This is stated as follows:

Theorem 5.3.3. *(***Integral Test***) Let $y = f(x)$ be a nice, positive, decreasing function defined on an interval $[k, \infty)$, where k is an integer. [That is: $\forall x \in [k, \infty), f(x) > 0$ and if $k \leq x_1 \leq x_2 < \infty$ then*

$f(x_1) \geq f(x_2)$.] *We let* $a_n = f(n)$ *for* $n = k, k+1, k+2, \ldots$ *Then*

$$\int_k^\infty f(x)dx \quad \text{converges (diverges)}$$

if and only if

$$\sum_{n=k}^\infty a_n \text{ converges (diverges).}$$

Note 5.3.4. For positive functions and positive series, respectively, the $\int_k^\infty f(x)dx$ and $\sum_{n=k}^\infty a_n$ diverges means it is equal to ∞

Remark 5.3.1. Whereas in calculus the Integral Test is mainly used to check the convergence or divergence of a positive series that satisfies the hypotheses of this criterion, here we use it in the converse way to check the convergence or divergence of an improper Riemann integral under these hypotheses. So we need to prove that the respective positive series converges or diverges. To this end we employ any different criterion that gives an answer for the series, among all those someone can find in books of advanced calculus or real analysis. Review these criteria one more time. For instance, we remark that under the conditions of the Integral Test the following three criteria are often very convenient.

Theorem 5.3.5. *(***Cauchy Positive Series Condensation Theorem***) Suppose that* $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$ *is a decreasing sequence of nonnegative numbers. Then*

$$\sum_{n=1}^\infty a_n \text{ converges (diverges)}$$

if and only if

$$\sum_{k=0}^\infty 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots \text{ converges (diverges).}$$

Theorem 5.3.6. *(**Absolute Root Test**. (**Cauchy**)) Consider a series of real numbers $\sum_{n=k}^{\infty} a_n$. Suppose the following limit exists or is ∞*

$$0 \leq \lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho \leq \infty$$

Then.

1. *If $0 \leq \rho < 1$, the series $\sum_{n=k}^{\infty} a_n$ converges absolutely and therefore it converges*
2. *If $1 < \rho \leq \infty$, the series $\sum_{n=k} a_n$ diverges.*
3. *If $\rho = 1$, the test is inconclusive.*

Theorem 5.3.7. *(**Absolute Ratio Test**. (**D′Alembert**)) Consider a series of real numbers $\sum_{n=k}^{\infty} a_n$. Suppose the following limit exists or is ∞*

$$0 \leq \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho \leq \infty$$

Then:

1. *If $0 \leq \rho < 1$, the series $\sum_{n=k}^{\infty} a_n$ converges absolutely and therefore it converges.*
2. *If $1 < \rho \leq \infty$, the series $\sum_{n=k}^{\infty} a_n$ diverges.*
3. *If $\rho = 1$, the test is inconclusive.*

Remark 5.3.2. The root and ratio tests as presented here are not stated in the most general form; one can find them in a good mathematical analysis book. In such a book these tests are stated in terms of the **liminf** and **limsup** of sequences of real numbers for achieving the most more general results. Study this material from a good mathematical analysis book.

Example 5.3.8.

1. The results presented in this example are straightforward, but because they are very useful we find them at times under the name **p-Test** . When combined with other tests it can answer a lot of questions on convergence or divergence of integrals rather easily.

   Since $\forall p \in \mathbb{R}$ the antiderivative of the function

   $$f(x) = x^{-p} = \frac{1}{x^p}, \quad x \in (0, \infty)$$

   is

   $$F(x) = \begin{cases} \frac{x^{-p+1}}{-p+1} + c & \text{if } p \neq 1 \\ \ln(|x|) + c & \text{if } p = 1 \end{cases}$$

   where c is an arbitrary constant, we obtain the following easy but useful results:

   Let $0 < k < \infty$ be a constant. Then:

   $$(1) \int_k^\infty \frac{dx}{x^p} = \begin{cases} \infty & \text{if } p \leq 1 \\ \frac{1}{k^{p-1}(p-1)} & \text{if } p > 1. \end{cases}$$

   $$(2) \int_0^k \frac{dx}{x^p} = \begin{cases} \frac{k^{1-p}}{1-p} = \frac{1}{k^{p-1}(1-p)} & \text{if } p < 1 \\ \infty & \text{if } p \geq 1 \end{cases}$$

   So, by both (1) and (2)

   $$\forall p \in \mathbb{R} \text{ we have: } \int_0^\infty \frac{dx}{x^p} = \infty.$$

   We now obtain the two byproducts:

(1) For any $a < b$ and $p < 1$ real constants

$$\int_a^b \frac{dx}{(x-a)^p} = \int_0^{b-a} \frac{dt}{t^p} = \int_a^b \frac{dx}{(b-x)^p} = \frac{(b-a)^{1-p}}{1-p}.$$

(2) For any a, b and p real constants

$$\int_a^\infty \frac{dx}{(x-a)^p} = \int_0^\infty \frac{dt}{t^p} = \int_{-\infty}^b \frac{dx}{(b-x)^p} = \infty.$$

An example of using the p-Test is in proving the convergence of the integral

$$\int_0^\infty \frac{\sin^2(x)}{x^2} dx = \int_0^1 \frac{\sin^2(x)}{x^2} dx + \int_1^\infty \frac{\sin^2(x)}{x^2} dx.$$

(The á-priori splitting of this integral is legitimate since the integrand function is positive.)

For the part

$$\int_0^1 \frac{\sin^2(x)}{x^2} dx$$

we observe

$$\lim_{x \to 0} \frac{\sin^2(x)}{x^2} = 1$$

a fact that makes the integral proper and therefore finite.

For the second part, we observe

$$\int_1^\infty \frac{\sin^2(x)}{x^2} dx < \int_1^\infty \frac{1}{x^2} dx = 1$$

and so this integral is convergent, by the **Comparison Test** 5.3.1 and the p-Test with $p = 2 > 1$

On the other hand, the integral

$$\int_0^\infty \frac{\sin^2(x)}{x^3}dx = \int_0^1 \frac{\sin^2(x)}{x^3}dx + \int_1^\infty \frac{\sin^2(x)}{x^3}dx$$

diverges because

$$\int_0^1 \frac{\sin^2(x)}{x^3}dx = \infty$$

This is so because near $x = 0$, $\frac{\sin^2(x)}{x^3} = \frac{\sin^2(x)}{x^2} \cdot \frac{1}{x}$ behaves like $b \cdot \frac{1}{x}$ for some constant $b > 0$. That is, we can find a constant $0 < k \leq 1$ such that for $0 < b = \frac{1}{2} < 1$ we have $\frac{\sin^2(x)}{x^2} \cdot \frac{1}{x} > \frac{1}{2} \cdot \frac{1}{x}$ for all $0 < x < k$ Then

$$\int_0^1 \frac{\sin^2(x)}{x^3}dx > \int_0^k \frac{1}{2} \cdot \frac{1}{x}dx = \frac{1}{2} \cdot [\ln(k) + \infty] = \infty$$

2. **Limit Comparison Test**. In this example we present a method analogous to the limit comparison test for positive series. We find the limit of the ratio of two positive functions as we approach a singularity or $\pm\infty$ and then we make an appropriate comparison of their integrals. We illustrate this with the following examples.

   (a) Prove that $\forall q \in \mathbb{R}$, $\int_1^\infty x^q e^{-x} dx$ converges.
   In $[1, \infty)$ we compare the positive function $f(x) := x^q e^{-x}$ with the positive function $g(x) := \frac{1}{x^2}$ by taking the limit $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} x^{q+2}e^{-x} = \lim_{x\to\infty} \frac{x^{q+2}}{e^x} = 0$ (by L' Hôpital's rule, e.g.).
   Therefore, there is a constant $k > 1$ such that $\frac{f(x)}{g(x)} < 1$ if $k \leq x < \infty$ or $0 < f(x) < g(x) < 1$ if $k \leq x < \infty$. But then

$$\int_k^\infty x^q e^{-x} < \int_k^\infty \frac{1}{x^2}dx = \left[\frac{-1}{x}\right]_k^\infty = \frac{1}{k}$$

Hence,

$$\int_1^\infty x^q e^{-x} dx = \int_1^k x^q e^{-x} dx + \int_k^\infty x^q e^{-x} dx < \text{ finite } + \frac{1}{k} < \infty$$

and therefore converges.

(b) Prove that $\forall q > -1, \int_0^1 x^q e^{-x} dx$ converges. (Notice that when $q \geq 0$ the integral is proper and so finite.)
When $-1 < q < 0$, the positive function $f(x) := x^q e^{-x}$ has singularity at $x = 0$ (it tends to ∞ as $x \longrightarrow 0^+$). So, for $-1 < q < 0$, we compare $f(x)$ with the positive function $g(x) := \frac{1}{x^q}$ by taking the limit

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} e^{-x} = 1$$

Therefore, there is a constant $0 < k < 1$ such that $\frac{f(x)}{g(x)} < 2$ if $0 < x \leq k$, i.e., $0 < f(x) < 2g(x)$ if $0 < x \leq k$. But, then $q + 1 > 0$ and

$$\int_0^k x^q e^{-x} dx < \int_0^k 2 \frac{1}{x^q} dx = 2 \left[\frac{x^{q+1}}{q+1} \right]_0^k = \frac{2k^{q+1}}{q+1}$$

Hence, when $-1 < q < 0$,

$$\begin{aligned} \int_0^1 x^q e^{-x} dx &= \int_0^k x^q e^{-x} dx + \int_k^1 x^q e^{-x} dx \\ &< \frac{2k^{q+1}}{q+1} + \text{ finite } < \infty \end{aligned}$$

and therefore converges. So,

$$\int_0^1 x^q e^{-x} dx \quad \text{converges for all } q > -1$$

(c) Prove that $\forall q \leq -1, \int_0^1 x^q e^{-x} dx (= \infty)$ diverges.

In$(0, 1]$ the positive function $f(x) := x^q e^{-x}$ has singularity at $x = 0$ (it tends to ∞ as $x \longrightarrow 0^+$). We compare it with the positive function $g(x) := x^q$ by taking the limit

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} e^{-x} = 1$$

Therefore, there is a constant $0 < k < 1$ such that $\frac{f(x)}{g(x)} > \frac{1}{2}$ if $0 < x \leq k$, i.e., $f(x) > \frac{1}{2} g(x)$ if $0 < x \leq k$. But then (by the previous example)

$$\int_0^k x^q e^{-x} dx \geq \int_0^k \frac{1}{2} \cdot x^q dx = \infty$$

Hence,

$$\int_0^k x^q e^{-x} dx = \infty$$

and therefore

$$\int_0^1 x^q e^{-x} dx = \int_0^k x^q e^{-x} dx + \int_k^1 x^q e^{-x} dx = \infty + \text{ finite } = \infty$$

diverges for all $q \leq -1$

3. We would like to prove that the integral

$$\int_2^\infty \frac{dx}{[\ln(x)]^{\ln(x)}}$$

converges.

One way to do this is to use the **Integral Test**, since same we easily observe the function

$$f(x) = \frac{1}{[\ln(x)]^{\ln(x)}}, \qquad \text{on} \qquad [2, \infty)$$

is positive for $x \geq 2$ and decreasing for $x \geq 3$. (Also, its limit is zero as $x \longrightarrow \infty$

So, to prove that this integral converges (is finite) we must prove that the positive series

$$\sum_{n=2}^{\infty} \frac{1}{[\ln(n)]^{\ln(n)}}$$

converges. This is done as follows:

We use the **Cauchy Condensation Theorem**, 5.3.5, all the hypotheses of which are satisfied. (Check this.) So, we must prove that the series

$$\sum_{k=2}^{\infty} \frac{2^k}{[\ln(2^k)]^{\ln(2^k)}} = \sum_{k=2}^{\infty} \frac{2^k}{[k\ln(2)]^{k\ln(2)}},$$

converges.

We prove that the latter series converges by using the **Absolute Root Test**. Indeed:

$$\begin{aligned} \rho := \lim_{k\to\infty} \sqrt[k]{|a_k|} &= \lim_{k\to\infty} \sqrt[k]{\frac{2^k}{[k\ln(2)]^{k\ln(2)}}} \\ &= \lim_{k\to\infty} \frac{2}{[k\ln(2)]\ln(2)} = 0 < 1. \end{aligned}$$

Hence, the last series converges and so the initial series converges too. Therefore, the given improper integral converges by the **Integral Test**.

Another way to prove that the given integral converges is the following: Using the substitution $t = \ln(x) \Longleftrightarrow x = e^t$ in the above integral we get

$$\int_2^{\infty} f(x)dx = \int_{\ln(2)}^{\infty} \frac{e^t}{t^t} dt$$

Then we observe that the function $g(t) = \frac{e^t}{t^t}$ is positive for $t \geq \ln(2)$ and decreasing for $t \geq 3$. (Also, its limit is zero as $x \longrightarrow \infty$.) Moreover, the series

$$\sum_{n=1}^{\infty} \frac{e^n}{n^n}$$

converges.

For the convergence of this series we can use, e.g., the **Root Test** to get

$$\lim_{n\to\infty} \sqrt[n]{\frac{e^n}{n^n}} = \lim_{n\to\infty} \frac{e}{n} = 0 < 1$$

So, the **Integral Test** applies and the improper integral converges.

4. With work similar to the previous example, we can prove that

$$\int_3^{\infty} \frac{dx}{[\ln[\ln(x)]]^{\ln(x)}}$$

converges.

But,

$$\int_3^{\infty} \frac{dx}{[\ln(x)]^{\ln[\ln(x)]}}$$

diverges.

Now we consider a nice function $y = f(x)$ with positive and negative values, defined in a set A. We say that the improper integral of $f(x)$ over $A \subseteq \mathbb{R}$ exists if $\int_A f(x)dx$ is a finite (real) value. However, in such a situation, we distinguish the following two cases and definitions:

Definition 43. We say that the improper integral of a nice function $f(x)$ over a set A exists or converges absolutely if

$$\int_A |f(x)|dx$$

is equal to a finite non-negative value.

Otherwise, $\int_A |f(x)|dx = \infty$ and then we say that the improper integral of $f(x)$ over the set A diverges absolutely.

Definition 44. We say that the improper inteqral of a nice function $f(x)$ over a set A converges conditionally if

$$\int_A f(x)dx$$

is equal to a finite real value (and so this improper integral exists), but it diverges absolutely.

Now we state the absolute convergence test which claims that absolute convergence implies convergence, but not vice-versa.

Theorem 5.3.9. *(***Absolute Convergence Test***) We consider a nice function $f(x)$ defined in a set A.*

(a) If

$$\int_A |f(x)|dx \quad \text{exists}$$

then

$$\int_A f(x)dx \text{ exists.}$$

(b) In Case (a) we also have the inequality

$$\left|\int_A f(x)dx\right| \leq \int_A |f(x)|dx$$

(c) The converse of this test is not true.

Proof. (a) The inequality $0 \leq |f(x)| - f(x) \leq 2|f(x)|$ is valid for all $x \in$ A. Now we apply the Non-negative Comparison Test (Theorem 5.3.1), and with a straightforward manipulation we obtain the result.

(b) The relation

$$\left|\int_A f(x)dx\right| \leq \int_A |f(x)|dx$$

is valid, because it is valid for any nice function and any closed interval $[p, q] \subseteq A$, as we have learnt in a calculus course.

(c) In the sequel we shall see several examples that disprove the converse.

□

Before we present concrete examples using the Absolute Convergence Test and/or other tests, we need to clear out some things about what is legitimate splitting of integrals into smaller parts. So, begin with the following example.

Example 5.3.10. Consider the improper integral

$$\int_a^\infty f(x)dx = \lim_{\substack{N\to\infty \\ \sigma\to a+}} \int_\sigma^N f(x)dx$$

being improper at both endpoints only. That is, $f(x)$ is nice in the open interval (a, ∞) and on any closed and bounded (finite) subinterval of (a, ∞) its integral is proper.

Then, if we pick any two fixed numbers b and c such that $a < b < c < \infty$, we can always write

$$\int_a^\infty f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^\infty f(x)dx$$

In this equality the first and the last summands are improper integrals only at the lower and the upper limit of integration, respectively. The middle summand is a proper integral.

The justification of such a splitting of this improper integral goes as follows: In taking the limits in this improper integral we do not lose anything by keeping $a < \sigma < b$ and $c < N < \infty$. Also, the following equality is always valid

$$\int_\sigma^N f(x)dx = \int_\sigma^b f(x)dx + \int_b^c f(x)dx + \int_c^N f(x)dx$$

because all the integrals of this equality are proper. Therefore,

$$\begin{aligned}
\int_a^\infty f(x)dx &= \lim_{\substack{N\to\infty \\ \sigma\to a^+}} \int_\sigma^N f(x)dx \\
&= \lim_{\substack{N\to\infty \\ \sigma\to a^+}} \left[\int_\sigma^b f(x)dx + \int_b^c f(x)dx + \int_c^N f(x)dx\right] \\
&= \lim_{\sigma\to a^+} \int_\sigma^b f(x)dx + \int_b^c f(x)dx + \lim_{N\to\infty} \int_c^N f(x)dx \\
&= \int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^\infty f(x)dx
\end{aligned}$$

For the same reasons, if $a < a_1 < a_2 < \ldots < a_n < a_{n+1} < \infty$, for any $n \in \mathbb{N}$, we can write

$$\int_a^\infty f(x)dx = \int_a^{a_1} f(x)dx + \sum_{k=1}^{n} \int_{a_k}^{a_{k+1}} f(x)dx + \int_{a_{n+1}}^{\infty} f(x)dx$$

Now we continue with a simple, nevertheless useful, lemma that tells us when and how we can break an improper integral into denumerable summations of appropriately chosen smaller pieces. As we know from calculus, it is always possible to break a proper integral into summations of countably (finitely or denumerably) many smaller proper integrals. But, even though we can write an improper integral as a summation of finitely many smaller parts, as this was done in the previous example, not all splittings into denumerable summations of smaller integrals are legitimate. The following lemma describes conditions under which these denumerable summations are valid.

Lemma 5.3.3.

(a) Let $y = f(x)$ be a function on the interval $[a, c)$, where $-\infty < a < c$ and $c \in \mathbb{R}$ or $c = \infty$. Consider any (strictly) increasing sequence $a = a_0 < a_1 < a_2 < \dots$ with $\lim_{n\to\infty} a_n = c$. Then:
If

(a_1) $\int_a f(x)dx$ *exists and it is equal to a real number L,*
or
(a_2) $f(x) \geq 0, \quad \forall x \in [a, c)$
or
(a_3) $f(x) \leq 0, \quad \forall x \in [a, c)$,

then

$$\int_a^c f(x)dx = \sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} f(x)dx$$

(b) Let $y = f(x)$ be a function on the interval $(c, b]$, where $c < b < \infty$ and $c \in \mathbb{R}$ or $c = -\infty$. Consider any (strictly) decreasing sequence $b = b_0 > b_1 > b_2 > \ldots$ with $\lim_{n\to\infty} b_n = c$. Then:

If

(b_1) $\int_c^b f(x)dx$ *exists and it is equal to a real number* L,

or

(b_2) $f(x) \geq 0, \quad \forall x \in (c, b]$,

or

(b_3) $f(x) \leq 0, \quad \forall x \in (c, b]$

then

$$\int_c^b f(x)dx = \sum_{n=0}^{\infty} \int_{b_{n+1}}^{b_n} f(x)dx$$

Proof. We shall prove **Case** ($\mathbf{a}$) only, for **Case** ($\mathbf{b}$) is just analogous. Also, **Subcase** ($\mathbf{a_3}$) is analogous to **Subcase** ($\mathbf{a_2}$) and so it suffices to prove only the **Subcases** ($\mathbf{a_1}$) and ($\mathbf{a}_2$).

In **Subcase** ($\mathbf{a_1}$), we assume that $\int_0^c f(x)dx = L$ for some real number L. Then

$$\int_a^c f(x)dx = \lim_{R\to c^-} \int_a^R f(x)dx = L.$$

Since the limit exists, any legitimate limiting process whatsoever gives the number L as value of the limit. Therefore, for any increasing sequence with limit $c\,(a_n \uparrow c$ as $n \to \infty)$ we have

$$\int_a^c f(x)dx = \lim_{n\to\infty} \int_{a=a_0}^{a_n} f(x)dx = L.$$

But, since we can break any proper integral into a finite sum of successive smaller proper integrals, we get, for $n \geq 1$

$$\int_{a=a_0}^{a_n} f(x)dx = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(x)dx$$

Then putting the last two equations together, we obtain

$$\begin{aligned}
\int_a^c f(x)dx &= \lim_{n\to\infty} \int_{a=a_0}^{a_n} f(x)dx \\
&= \lim_{n\to\infty} \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(x)dx \\
&= \sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} f(x)dx = L
\end{aligned}$$

In **Subcase** $(\mathbf{a}_2)$, we assume that $f(x) \geq 0, \forall x \in [a, c)$. If it happens that $\int_a^c f(x)dx$ exis, then we invoke the previous subcase and the proof is over. If the integral does not exist, since $f(x) \geq 0$ this means that

$$\int_a^c f(x)dx = \lim_{R\to c^-} \int_a^R f(x)dx = +\infty$$

Now, given any R such that $a < R < c$, since $\lim\limits_{n\to\infty} a_n = c$, we can pick a term a_{n+1} of the sequence such that $R \leq a_{n+1}$. Then by the non-negativity of the function, we have

$$\begin{aligned}
\int_a^R f(x)dx &\leq \int_a^R f(x)dx + \int_R^{a_{n+1}} f(x)dx \\
&= \int_a^{a_{n+1}} f(x)dx \\
&= \sum_{k=0}^{n} \int_{a_k}^{a_{k+1}} f(x)dx.
\end{aligned}$$

Also, for any given term a_{n+1} of the sequence, where $n =\in \mathbb{N}$, we can pick a real R such that $a < R \leq a_{n+1}$ to obtain again a similar inequality.

Since

$$\lim_{R\to c^-} \int_a^R f(x)dx = \infty$$

$a < R \leq a_{n+1}$ and $f(x) \geq 0,$ we get

$$\infty = \lim_{R\to c^-} \int_a^R f(x)dx \leq \lim_{n\to\infty} \sum_{k=0}^{n} \int_{a_k}^{a_{k+1}} f(x)dx = \sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} f(x)dx.$$

Therefore,

$$\int_a^c f(x)dx = \lim_{R\to c^-} \int_a^R f(x)dx = \sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} f(x)dx = \infty$$

□

Example 5.3.11.

1. The previous lemma does not apply in either of the two cases of $\int_a^c f(x)dx$ or $\int_c^b f(x)dx$ when these improper integrals do not exist.

For instance, in Example 5.1.3 (d) we have seen that the integral

$$\int_0^\infty \sin(x)dx \text{ does not exist.}$$

If we now let $a_n = 2n\pi$ for $n = 0, 1, 2, \ldots$, which satisfies all the requirements of the lemma, then we get

$$\sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} \sin(x)dx = \sum_{n=0}^{\infty} \int_{2n\pi}^{2(n+1)\pi} \sin(x)dx = \sum_{n=0}^{\infty} 0 = 0$$

even though the integral itself does not exist

With $a_n = n\pi$ for $n = 0, 1, 2, \ldots$, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} \sin(x)dx &= \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \sin(x)dx \\ &= \sum_{n=0}^{\infty} 2(-1)^n \\ &= \text{ does not exist} \end{aligned}$$

an answer different from the one found before.

(For your own practice, find some other sequences $(a_n)_{n\in\mathbb{N}}$ that satisfy the requirements of the lemma and yield other values for the respective infinite summation in this example.)

2. The improper integral

$$\int_0^\infty \frac{\sin(x)}{x^2+1}dx$$

is absolutely convergent and therefore convergent.

To show this, we notice

$$\int_0^\infty \left|\frac{\sin(x)}{x^2+1}\right| dx < \int_0^\infty \frac{1}{x^2+1} dx = \frac{\pi}{2}$$

So,

$$\frac{-\pi}{2} < \int_0^\infty \frac{\sin(x)}{x^2+1} dx < \frac{\pi}{2}.$$

3. Let

$$f(x) = \frac{\sin(x)}{n+1} \quad \text{for} \quad n\pi \le x \le (n+1)\pi, n = 0, 1, 2, \ldots .$$

With the help of Lemma 5.3.3, **Case** ($\mathbf{a}_2$), applied to the non-negative function $|f(x)|$, we obtain

$$\begin{aligned} \int_0^\infty |f(x)| dx &= \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \frac{|\sin(x)|}{n+1} dx \\ &= 2\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right) = \infty \end{aligned}$$

So, the improper integral $\int_0^\infty f(x)dx$ diverges absolutely, and we cannot claim anything about its conditional convergence yet.

Since we do not know that this integral converges (exists), we cannot apply Lemma 5.3.3 at this point in order to say that

$$\int_0^\infty f(x)dx \stackrel{?}{=}$$

$$\sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \frac{\sin(x)}{n+1} dx = 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right) = 2\ln(2)$$

To prove that it converges (conditionally) we must use a different argument. If convergence is established first, then we can use Lemma 5.3.3 in order to claim that this equation is correct and the value of the improper integral is $2\ln(2)$.

So to prove that this integral converges, we proceed as follows: For any $R > 0$ there is an integer $k \geq 0$ such that $k\pi \leq R < (k+1)\pi$. Then, $(R \longrightarrow \infty) \Longleftrightarrow (k \longrightarrow \infty)$ and by definition we have

$$\begin{aligned}
&\int_0^\infty f(x)dx \\
&\stackrel{def}{=} \lim_{R\to\infty} \int_0^R f(x)dx = \lim_{R\to\infty} \left[\int_0^{(k+1)\pi} f(x)dx - \int_R^{(k+1)\pi} f(x)dx\right] \\
&= \lim_{R\to\infty} \left\{2\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^k \frac{1}{k+1}\right] - \int_R^{(k+1)\pi} f(x)dx\right\}
\end{aligned}$$

But, the two partial limits inside the bracket exist. For the first one, we know

$$\lim_{R\to\infty} 2\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^k \frac{1}{k+1}\right] = 2\ln(2)$$

For the second one, we have

$$\lim_{R\to\infty} \left[\int_R^{(k+1)\pi} f(x)dx\right] = 0$$

since we easily observe that

$$0 < \left|\int_R^{(k+1)\pi} f(x)dx\right| \leq \int_{k\pi}^{(k+1)\pi} |f(x)|dx = \frac{2}{k+1} \longrightarrow 0,$$
$$\text{as } k \longrightarrow \infty \Longleftrightarrow R \to \infty.$$

Since these two partial limits exist, we can take their difference to obtain

$$\int_0^{\infty} f(x)dx \stackrel{def}{=} \lim_{R\to\infty}\int_0^R f(x)dx =$$

$$2\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots\right)-0 = 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 2\ln(2)$$

Hence, this improper integral converges conditionally to the number $2\ln(2)$

4. Let

$$g(x) = \frac{\sin(x)}{(n+1)^2} \qquad \text{for} \qquad n\pi \le x \le (n+1)\pi, n = 0, 1, 2, \dots$$

For the absolute convergence, we can apply Case ($\mathbf{a}_2$) of Lemma 5.3.3 to the non-negative function $|g(x)|$ to find

$$\begin{aligned}\int_0^{\infty} |g(x)|dx &= \sum_{n=0}^{\infty}\int_{n\pi}^{(n+1)\pi}\frac{|\sin(x)|}{(n+1)^2}dx \\ &= 2\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots\right) \\ &= 2\cdot\frac{\pi^2}{6} \\ &= \frac{\pi^2}{3}\end{aligned}$$

Hence, this improper integral converges absolutely, and so it converges.

Now, by **Case** ($\mathbf{a}_3$) of the Lemma 5.3.3 it is legitimate to say that

$$
\begin{aligned}
\int_0^{\infty} g(x)dx &= \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{\sin(x)}{(n+1)^2} dx \\
&= 2\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right) \\
&= 2 \cdot \frac{\pi^2}{12} \\
&= \frac{\pi^2}{6}
\end{aligned}
$$

This is the actual finite value of this integral.

The ideas in Example 5.3.11(3) motivate us to state the following useful lemma about convergence (existence) and estimation or actual evaluation of improper integrals. Its proof is omitted as analogous to the series of arguments presented in the solution of this example. (You can write it out for practice.) This lemma can also be used to justify the splitting of improper integrals into an infinite summation of smaller parts, and so it should be viewed together with Lemma 5.3.3

Lemma 5.3.4.

(a) Let $y = f(x)$ be a function on an interval $[a, c)$, where $-\infty < a < c$ and $c \in \mathbb{R}$ or $c = \infty$. Consider any (strictly) increasing sequence $a = a_0 < a_1 < a_2 < \dots$ with $\lim_{n\to\infty} a_n = c$.

We assume:

(a_1)

$$\lim_{n\to\infty} \sum_{k=0}^{n} \int_{a_n}^{a_{n+1}} f(x)dx = l, \qquad \textit{with} \qquad -\infty \le l \le \infty$$

(a_2) *For any real number* R *such that* $a < R < c$ *and the unique* $k \in \mathbb{N}$ *such that* $a_k \leq R < a_{k+1}$*[*k *depends on* R *and is unique since the sequence* (a_n) *is strictly increasing] we have:*

$$\lim_{R \to c^-} \int_R^{a_{k+1}} f(x)dx = 0.$$

Then,

$$\int_a^c f(x)dx = \sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} f(x)dx = l.$$

(b) Let $y = f(x)$ *be a function on an interval* $(c, b]$*, where* $c < b < \infty$ *and* $c \in \mathbb{R}$ *or* $c = -\infty$*. Consider any (strictly) decreasing sequence* $b = b_0 > b_1 > b_2 > \dots$ *with* $\lim\limits_{n \to \infty} b_n = c$

We assume:

(b_1)

$$\lim_{n \to \infty} \sum_{k=0}^{n} \int_{b_{n+1}}^{b_n} f(x)dx = l, \qquad \textit{with } -\infty \leq l \leq \infty$$

(b_2) *For any real number* R *such that* $c < R < b$ *and the unique* $k \in \mathbb{N}$ *such that* $b_{k+1} < R \leq b_k$*[*k *depends on* R *and is unique since the sequence* (b_n) *is strictly decreasing] we have:*

$$\lim_{R \to c^+} \int_{b_{k+1}}^{R} f(x)dx = 0$$

Then,

$$\int_c^b f(x)dx = \sum_{n=0}^{\infty} \int_{b_{n+1}}^{b_n} f(x)dx = l$$

Remark 5.3.5. Given that in this lemma the $\int_{a_n}^{a_{n+1}} f(x)dx$ is assumed to be proper and therefore finite and R is any number in the interval $[a_n, a_{n+1})$, we can only assume that $\lim\limits_{R \to c^-} \int_R^{a_{n+1}} f(x)dx = 0$ and not another finite number, since for R close enough to a_{n+1} we can make this partial integral as close to zero as we wish.

Remark 5.3.6. Notice that assumotion (a_2) of this lemma fails in Example 5.1.3 (4). As we have seen in that example and in Example 5.3.11 (1), the integral $\int_0^\infty \sin(x)dx$ does not exist.

We conclude this section with the very important and powerful criterion of Cauchy. We interpret it in the following way:

Theorem 5.3.12. *(**Cauchy Test**) Let $y = f(x)$ be a nice function on $[a, c)$, where $a \in \mathbb{R}$ and $c \in \mathbb{R}$ with $a < c$, or $c = \infty$. Consider the following three statements:*

(a) $\int_a^c f(x)dx$ *converges.*

(b) $\forall \epsilon > 0, \exists N \in \mathbb{R} : a \leq N < c$ *such that* $\forall p \in \mathbb{R}$ *and* $\forall q \in \mathbb{R}$ *such that* $N \leq p, q < c$, *we have*

$$\left| \int_p^q f(x)dx \right| < \epsilon$$

(c) $\forall r \in [a, c)$, *the integral* $\int_a^r f(x)dx$ *exists.*

Then we have:

(I) (a) *implies (b)*

(II) If (c) *holds, then the converse of* (I) *is true, i.e.,* (b) *implies* (a)*.*

Remark 5.3.7. We have analogous results for $\int_c^b f(x)dx$, on $(c, b]$, with $c < b$ in $\mathbb{R}$, or $c = -\infty$. (Write down these results explicitly, for practice.)

Remark 5.3.8. Hypothesis (c) is needed in Part (II) as seen in the the example that follows the proof. It is valid when $f(x)$ is continuous, or bounded in $[a, c)$, and in some other situations. In most applications, we are interested in using Part (II).

Proof (I) We assume

$$\int_a^c f(x)dx = \lim_{M \to c^-} \int_a^M f(x)dx = L$$

exists as a real finite value L

Then we let

$$F(M) = \int_a^M f(x)dx, \quad \forall M \in [a, c)$$

By our assumption, the function $F(M)$ is well defined on $[a, c)$ and it is continuous in the variable M. (From calculus, we already know that the definite integral of a nice function is continuous with respect to its upper limit.) Also,

$$\lim_{M \to c^-} F(M) = L$$

Now, $\forall \epsilon > 0$, we consider $\frac{\epsilon}{2} > 0$ and we use the analytical definition of the existence of a limit to claim that:

$\exists N : a \leq N < c$ such that $\forall M : N \leq M < c$ the inequality $|F(M) - L| < \dfrac{\epsilon}{2}$ is true.

Then, for any p and $q : N \leq p, q < c$ we get

$$|F(q) - F(p)| = |F(q) - F(p) + L - L|$$

$$\leq |F(q) - L| + |F(p) - L|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since

$$|F(q) - F(p)| = \left| \int_a^q f(x)dx - \int_a^p f(x)dx \right| = \left| \int_p^q f(x)dx \right|$$

we obtain the claim:

$\forall \epsilon > 0, \exists N, a \leq N < c$ such that, for any p and q in $\mathbb{R}$:

$$N \leq p, q < c \Longrightarrow \left| \int_p^q f(x)dx \right| < \epsilon$$

(II) By hypothesis (c) the function $F(r)$, as defined above in the proof of (I), is well defined on $[a, c)$

Then, the hypotheses of this converse implication are translated as follows:

$$\forall \epsilon > 0, \exists N, a \leq N < c \text{ such that, for any } p \text{ and } q :$$
$$N \leq p, q < c \Longrightarrow |F(q) - F(p)| < \epsilon$$

By the Cauchy General Criterion for convergence in the real line we readily obtain that $\lim_{M \to \infty} F(M)$ exists as a finite real value.

Therefore, $\int_a^\infty f(x)dx$ exists, i.e., it is a finite real value.

Example 5.3.13. 1. In the Cauchy Test, above, for the converse of

(I) in Part (II), hypothesis (c) is necessary. For instance, we let

$$f(x) = \begin{cases} 5, & \text{if} \quad x = 0 \\ \frac{1}{x}, & \text{if} \quad 0 < x < 1 \\ 0, & \text{if} \quad x \geq 1 \end{cases}$$

Obviously, $f(x)$ is a nice piecewise continuous function on $[0, \infty)$, and

$$\int_0^\infty f(x)dx = \ln(1) - \ln\left(0^+\right) = 0 - (-\infty) = \infty$$

i.e., (a) is false.

But (b) is true since for any $\epsilon > 0$ we can pick $N \geq 1$. This happens because (c) fails, and so the function $F(r)$, in the above proof, is not well defined for all $r \in [0, \infty)$

Another example is

$$g(x) = \begin{cases} 5, & \text{if} \quad x = 0 \\ \frac{1}{x}, & \text{if} \quad -1 < x \neq 0 < 1 \\ 0, & \text{if} \quad x \geq 1 \end{cases}$$

$g(x)$ is a nice piecewise continuous function on $[-1, \infty)$, and

$$\int_{-1}^\infty g(x)dx = -\infty + \infty = \text{ does not exist}$$

i.e., (a) is false. But again $(\mathbf{b})$ is true since for any $\epsilon > 0$ we can pick $N \geq 1$

2. In this example, we will prove the following two important results:

   (a) $\int_0^\infty \frac{\sin(x)}{x} dx$ exists (converges to a finite real value).

   (b) This integral does not converge absolutely and so it converges conditionally.

Proof. (a): The continuous function $f(x) = \frac{\sin(x)}{x}$ in $(0, \infty)$ can be continuously extended to $x = 0$ by letting $f(0) = 1$, since $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. So, the integral (a) is improper only because it is taken over an unbounded interval.

Using integration by parts, we get that for any $0 < p < q$,

$$\int_p^q \frac{\sin(x)}{x} dx = \int_p^q \frac{-1}{x} d\cos(x) = \frac{\cos p}{p} - \frac{\cos q}{q} - \int_p^q \frac{\cos(x)}{x^2} dx$$

Thus

$$\begin{aligned}\left|\int_p^q \frac{\sin(x)}{x} dx\right| &\leq \frac{1}{p} + \frac{1}{q} + \int_p^q \frac{|\cos(x)|}{x^2} dx \\ &\leq \frac{1}{p} + \frac{1}{q} + \int_p^q \frac{1}{x^2} dx \\ &= \frac{1}{p} + \frac{1}{q} + \left[\frac{-1}{x}\right]_p^q \\ &= \frac{2}{p} \to 0, \qquad \text{as } p \to \infty\end{aligned}$$

Then $\forall \epsilon > 0$, if we pick any $q > p > \frac{2}{\epsilon}$, we get $\left|\int_p^q \frac{\sin(x)}{x} dx\right| < \epsilon$

Since here condition (c) of the Cauchy Test is valid, $\int_0^\infty \frac{\sin(x)}{x} dx$ exists as a finite value, by the Cauchy Test,

(b): This integral does not converge absolutely. Indeed, $|f(x)| \geq 0$, and so by Lemma 5.3.3 we have

$$\begin{aligned}\int_0^\infty |f(x)|dx &= \int_0^\infty \frac{|\sin(x)|}{x}dx \\ &= \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \frac{|\sin(x)|}{x}dx\end{aligned}$$

Now for all $n = 0, 1, 2, 3, \ldots$, we have

$$\begin{aligned}\int_{n\pi}^{(n+1)\pi} \frac{|\sin(x)|}{x}dx &\geq \frac{1}{(n+1)\pi}\int_{n\pi}^{(n+1)\pi} |\sin(x)|dx \\ &= \frac{2}{(n+1)\pi}\end{aligned}$$

So,

$$\begin{aligned}\int_0^\infty |f(x)|dx &= \int_0^\infty \frac{|\sin(x)|}{x}dx \\ &\geq \sum_{n=0}^\infty \frac{2}{(n+1)\pi} \\ &= \frac{2}{\pi}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right) \\ &= \infty.\end{aligned}$$

Therefore,

$$\int_0^\infty \frac{|\sin(x)|}{x}dx = \infty$$

$\square$

Remark 5.3.9. By observing that the function $f(x) = \frac{\sin(x)}{x}$ is an even and continuous function over all $\mathbb{R}$, we also obtain that

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx = 2 \int_{0}^{\infty} \frac{\sin(x)}{x} dx$$

Therefore, this improper integral over all $\mathbb{R}$ converges conditionally, too.

Remark 5.3.10. The same results of conditional convergence can be obtained for the improper integrals

$$\int_{0}^{\infty} \frac{\sin(\beta x)}{x} dx = \operatorname{sign}(\beta) \cdot \int_{0}^{\infty} \frac{\sin(u)}{u} du$$

and

$$\int_{-\infty}^{\infty} \frac{\sin(\beta x)}{x} dx = \operatorname{sign}(\beta) \cdot \int_{-\infty}^{\infty} \frac{\sin(u)}{u} du$$

for any real constant $\beta \neq 0$. (If $\beta = 0$ the integrals are obviously zero.)

The equality is obtained by letting $u = \beta x$ and the definition

$$\operatorname{sign}(\beta) = \begin{cases} +1, & \text{if} \quad \beta > 0 \\ -1, & \text{if} \quad \beta < 0 \end{cases}$$

Remark 5.3.11. Lemma 5.3.4 can also be used to prove convergence, except the Cauchy Test is more efficient.

Example 5.3.14. The Fresnel integral $\int_0^\infty \sin\left(x^2\right) dx$ converges conditionally.

We have

$$\int_{0}^{\infty} \sin\left(x^2\right) dx \overset{def}{=} \lim_{0<R\to\infty} \int_{0}^{R} \sin\left(x^2\right) dx$$

We want to show that this limit exists. That is, every limiting process gives the same finite answer.

We let $x^2 = u$ or $x = \sqrt{u}$ and $dx = \frac{du}{2\sqrt{u}}$. We notice that $\lim\limits_{u \to 0} \frac{\sin(u)}{\sqrt{u}} = 0$, and so we do not introduce any singularity at $x = 0$. Then, for any $q > p > 0$, using integration by parts, we get

$$\begin{aligned}
\int_p^q \sin\left(x^2\right) dx &= \int_{p^2}^{q^2} \sin(u) \frac{du}{2\sqrt{u}} \\
&= \frac{1}{2} \int_{p^2}^{q^2} \frac{1}{\sqrt{u}} d[-\cos(u)] \\
&= \frac{-1}{2} \left[\frac{\cos(u)}{\sqrt{u}}\right]_{p^2}^{q^2} + \frac{1}{2} \int_{p^2}^{q^2} \cos(u) \cdot d\left(u^{\frac{-1}{2}}\right) \\
&= \frac{-1}{2} \left[\frac{\cos\left(q^2\right)}{q} - \frac{\cos\left(p^2\right)}{p}\right] - \frac{1}{4} \int_{p^2}^{q^2} \frac{\cos(u)}{u^{\frac{3}{2}}} du
\end{aligned}$$

Therefore, by the properties of absolute value combined with inequalities and integrals and the fact that $|\cos(u)| \leq 1, \forall u \in \mathbb{R}$, we have

$$\begin{aligned}
\left|\int_p^q \sin\left(x^2\right) dx\right| &\leq \frac{1}{2} \left[\frac{1}{q} + \frac{1}{p}\right] + \frac{1}{4} \int_{p^2}^{q^2} \frac{1}{u^{\frac{3}{2}}} du \\
&= \frac{1}{2} \left[\frac{1}{q} + \frac{1}{p}\right] - \frac{1}{2} \left[\frac{1}{\sqrt{u}}\right]_{p^2}^{q^2} \\
&= \frac{1}{p}.
\end{aligned}$$

Now, as in the previous example, $\forall \epsilon > 0$ we choose $p > \frac{1}{\epsilon}$ to get that $\forall q > p > \frac{1}{\epsilon}$ to guarantee the validity of the inequality

$$\left|\int_p^q \sin\left(x^2\right) dx\right| < \epsilon$$

Since for any $r > 0$ the integral $\int_0^r \sin\left(x^2\right) dx$ exists, by the Cauchy Test

$$\int_0^\infty \sin\left(x^2\right) dx$$

converges to a finite real value.

Now we prove that the integral diverges absolutely. Again we let $x^2 = u$ to obtain

$$\begin{aligned}
\int_0^\infty \left|\sin\left(x^2\right)\right| dx &= \frac{1}{2}\int_0^\infty \left|\frac{\sin(u)}{\sqrt{u}}\right| du \\
&> \frac{1}{2}\int_0^1 \frac{|\sin(u)|}{\sqrt{u}} du + \frac{1}{2}\int_1^\infty \frac{|\sin(u)|}{u} du \\
&= \text{ finite } + \infty = \infty,
\end{aligned}$$

by the analysis of the previous example! (All the steps here are directly justified by the definitions and the fact that we work with a non-negative function.)

So, the Fresnel integral $\int_0^\infty \sin\left(x^2\right) dx$ converges but diverges absolutely. Therefore, it converges conditionally.

With parallel work and analogous adjustments, we also obtain that the other Fresnel integral

$$\int_0^\infty \cos\left(x^2\right) dx$$

converges conditionally.

EXERCISES

1. Give examples of functions with discontinuities of the three types that we have stated in the non-standard Definition 42.

2. (i) Prove:

$$\text{(a)} \quad \int_0^\infty \frac{dx}{1+x^3} = \frac{2\pi\sqrt{3}}{9},$$

$$\text{(b)} \quad \int_0^\infty \frac{dx}{1+x^4} = \frac{\pi\sqrt{2}}{4}.$$

Hint: You need partial fractions and the integral rules with the natural logarithm and arc-tangent.]

(ii) Use the results of the previous part and apply integration by parts to prove:

$$\text{(a)} \quad \int_0^\infty \frac{dx}{\left(1+x^3\right)^2} = \frac{4\pi\sqrt{3}}{27},$$

$$\text{(b)} \quad \int_0^\infty \frac{dx}{\left(1+x^4\right)^2} = \frac{3\pi\sqrt{2}}{16}.$$

3. The Fresnel cosine integral is

$$\int_0^\infty \cos\left(x^2\right) dx$$

Prove that it exists but diverges absolutely.

In the problems (4-11) below, check in any possible way the existence or non-existence of the given improper integrals.

4. (a) $\int_0^\infty \frac{dx}{\sqrt{1+x^3}}$,

(b) $\int_0^\infty \frac{dx}{\sqrt{1+x^4}}$.

5. (a) $\int_0^\infty \frac{\sin^2(x)}{x} dx$,

(b) $\int_0^\infty \frac{\sin^2(x)}{x^2} dx$.

6. (a) $\int_0^\infty \frac{\sin^3(x)}{x} dx$,

(b) $\int_0^\infty \frac{\sin^3(x)}{x^2} dx$,

(c) $\int_0^\infty \frac{\sin^3(x)}{x^3} dx$.

7. (a) $\int_0^\infty \frac{e^{-x}\sin(x)}{x} dx$,

(b) $\int_0^\infty \frac{x dx}{\sqrt{1+x^3}}$.

8. $\int_0^\infty \frac{\sin(ax)}{e^{bx}-1} dx$, where $a \in \mathbb{R}$ and $b > 0$ constants.

9. (a) $\int_2^\infty \frac{dx}{[\ln(x)]^p}$,

(b) $\int_1^\infty \frac{\sin(x)}{x^p} dx$,

(c) $\int_1^\infty \frac{\cos(x)}{x^p} dx$.

where $p > 0$ constant.

10. $\int_{-\infty}^\infty e^{\frac{-(x-\mu)^2}{\sigma^2}} dx$, where $\mu \in \mathbb{R}$ and $\sigma \neq 0$ constants.

11. (a) $\int_1^{10} \frac{dx}{\sqrt{x-1}}$,

(b) $\int_0^1 \frac{dx}{\sqrt[3]{x-1}}$

12. Prove that the integrals

(a) $\int_0^\infty \sin(e^x)\, dx$,

(b) $\int_0^\infty \cos(e^x)\, dx$ and

(c) $\int_0^\infty \frac{\sin(x)}{\sqrt{x^2+1}} dx$

(d) $\int_0^\infty \frac{\cos(x)}{\sqrt{x^2+1}} dx$

do not converge absolutely but only conditionally.

[Hint: For the first two, use $u = e^x$ and the Cauchy Test,]

13. Prove that $\forall n \geq 0$ integer the following integrals exist:

    (a) $\int_{-\infty}^{\infty} x^n e^{-x^2} dx$,

    (b) $\int_{-\infty}^{\infty} x^n e^{-|x|} dx$ If the integer is odd, then their values are zero.

14. Prove that for every $n \in \mathbb{N}$ the inequality $\sqrt{\frac{n}{2}} < \sqrt[n]{n!} < \frac{n+1}{2}$ holds and $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$

15. If (a_n) with $n \in \mathbb{N}$ is a sequence of positive numbers and the limits

$$\lim_{n\to\infty} \sqrt[n]{a_n} \text{ and } \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

    exist, then prove that they are equal.

5.4 Calculus Techniques

Here we present some Real Analysis Techniques for the computation of the precise value of some important improper integrals. Here we introduce the very useful **Euler – Poisson – Gauß Integral**.

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2\int_{0}^{\infty} e^{-x^2} dx = 2\int_{-\infty}^{0} e^{-x^2} dx = \sqrt{\pi}. \qquad (5.1)$$

By Example 5.3.2 (1), it exists, that is, it equals to a positive finite value. The first two equalities are due to the fact that the function $f(x) = e^{-x^2}$ is positive and even over $(-\infty, \infty) = \mathbb{R}$.

This integral is very useful in various applications in mathematics, physics, engineering, probability and statistics. Many times, it is calculated in a multi-variable calculus course. To find its precise value, we work as follows:

(a) We evaluate the double integral

$$\iint\limits_{\overline{D(0,a)}} e^{-\left(x^2+y^2\right)} dxdy$$

where $\overline{D(0,a)} = \left\{(x,y) \mid x^2+y^2 \leq a^2\right\}$ is the closed (circular) disc of center $(0,0)$ and radius $a > 0$.

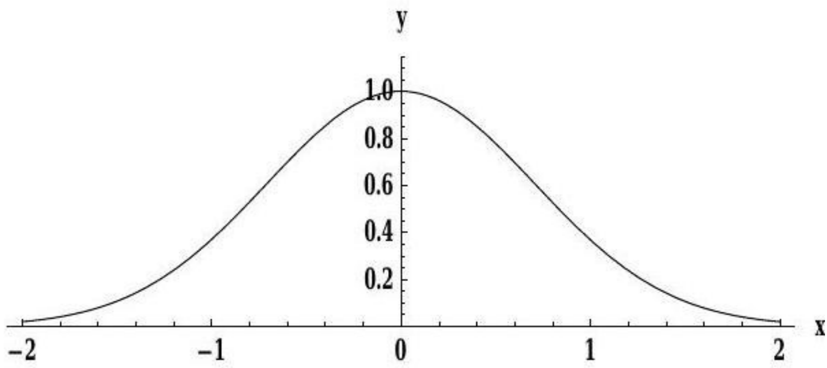

Figure 5.4: Function $y = e^{-x^2}$

It is usually more convenient to switch to polar coordinates when we work with circular discs with center the origin. So.

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad x^2+y^2 = r^2, \quad dxdy = rdrd\theta$$

and

$$\overline{D(0,a)} = \{(r,\theta) \mid 0 \leq r \leq a, 0 \leq \theta \leq 2\pi\}.$$

Hence the integral in polar coordinates r, θ is

$$\iint\limits_{D(0,a)} e^{-\left(x^2+y^2\right)} dxdy = \int_0^{2\pi} \int_0^a e^{-r^2} r dr d\theta$$
$$= 2\pi \left[\frac{-e^{-r^2}}{2}\right]_0^a$$
$$= \pi\left(1 - e^{-a^2}\right).$$

(b) Take the limit as $a \to \infty$ to find

$$\int_{\mathbb{R}^2}\int e^{-\left(x^2+y^2\right)} dxdy = \lim_{a\to\infty} \pi\left(1 - e^{-a^2}\right) = \pi(1-0) = \pi.$$

[Since the function $e^{-\left(x^2+y^2\right)}$ is continuous and positive in $\mathbb{R}^2$, advanced integration theory proves that it is integrable and for any nonnegative integrable function any legitimate limit process yields the unique non-negative real or $+\infty$ value of its integral.

(c) Now we view the integral

$$\int_{\mathbb{R}^2}\int e^{-\left(x^2+y^2\right)} dxdy$$

as the limit of integrals over the rectangles $R_a = [-a, a] \times [-a, a]$, as $0 < a \to \infty$, i.e,

$$\lim_{a\to\infty} \int_{R_a}\int e^{-\left(x^2+y^2\right)} dxdy = \pi,$$

Or

$$\lim_{a\to\infty} \int_{-a}^{a}\int_{-a}^{a} e^{-x^2} e^{-y^2} dxdy = \pi.$$

Then

$$\lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx \cdot \int_{-a}^{a} e^{-y^2}\,dy\right) = \pi.$$

(d) In Example 5.3.2 (1) we have proved that

$$\lim_{a\to\infty}\int_{-a}^{a} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx \quad \text{exists.}$$

So, the last equation can be rewritten as

$$\left(\lim_{a\to\infty}\int_{-a}^{a} e^{-x^2}\,dx\right) \cdot \left(\lim_{a\to\infty}\int_{-a}^{a} e^{-y^2}\,dy\right) = \pi.$$

These two limits are the same, and so

$$\left(\lim_{a\to\infty}\int_{-a}^{a} e^{-x^2}\,dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi.$$

Finally, since $\int_{-\infty}^{\infty} e^{-x^2}\,dx > 0$ as an integral of a positive function, we can take square roots of both sides of the last equality to get the result

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Remark 5.4.1. Since $f(x) = e^{-x^2}$ is an even function over $\mathbb{R}$, we have that

$$\int_{0}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{0} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Important Note : If a function of two variables is non-negative or nonpositive (i.e., does not change sign), then its double integral over a domain can be manipulated in any way and iterated in any order without affecting the final result. Otherwise, we would need the integral of its absolute value over its domain to exist or some other conditions.

In the above example, the function $e^{-\left(x^2+y^2\right)}$ in $\mathbb{R}^2$ is positive. Therefore, our manipulations and iterations do not alter the existence and the uniqueness of the final answer.

Since the integral $\int_0^x e^{-t^2} dt$ cannot be found in closed form, we define, by means of the integral computed here, and use the following two standard functions in theory and application.

(1) **The error function** $\operatorname{erf}(x)$

$$\forall x \in \mathbb{R}, \quad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Notice

$$\lim_{x \to \infty} \operatorname{erf}(x) = 1$$

This function is very important to application and has been tabulated.

(2) **The complementary error function** $\operatorname{erfc}(x)$

$\forall x \in \mathbb{R}, \quad \operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$

Notice

$$\lim_{x \to \infty} \operatorname{erfc}(x) = 0$$

EXERCISES

In problems 1-10 find the precise values of the given improper integrals.

1. $\int_{-\infty}^{\infty} e^{-3x^2} dx, \quad \int_{-\infty}^{0} e^{-3x^2} dx$ and $\int_0^\infty e^{-3x^2} dx.$
2. $\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{\sigma^2}} dx, \quad \int_{\mu}^{\infty} e^{-\frac{(x-\mu)^2}{\sigma^2}} dx$ and $\int_{-\infty}^{\mu} e^{-\frac{(x-\mu)^2}{\sigma^2}} dx$. where μ and $\sigma \neq 0$ are real constants.

3. $\int_0^\infty xe^{-x^2}dx, \quad \int_{-\infty}^0 xe^{-x^2}dx$ and $\int_{-\infty}^\infty xe^{-x^2}dx$.

4. $\int_0^\infty x^2e^{-x^2}dx, \quad \int_{-\infty}^0 x^2e^{-x^2}dx$ and $\int_{-\infty}^\infty x^2e^{-x^2}dx$.

5. $\int_{-\infty}^\infty x^3e^{-x^2}dx, \quad \int_0^\infty x^3e^{-x^2}dx$ and $\int_{-\infty}^0 x^3e^{-x^2}dx$.

6. $\int_0^\infty xe^{-x}dx, \quad \int_{-\infty}^0 xe^{-x}dx$ and $\int_{-\infty}^\infty xe^{-x}dx$.

7. $\int_0^\infty x^2e^{-x}dx, \quad \int_{-\infty}^\infty x^2e^{-x}dx$ and $\int_{-\infty}^0 x^2e^{-x}dx$.

8. $\int_0^\infty x^3e^{-x}dx, \quad \int_{-\infty}^0 x^3e^{-x}dx$ and $\int_{-\infty}^\infty x^3e^{-x}dx$.

9. $\int_{-\infty}^\infty xe^{-x^4}dx, \quad \int_{-\infty}^0 xe^{-x^4}dx$ and $\int_0^\infty xe^{-x^4}dx$. Hint: Let $u = x^2$, etc.]

10. $\int_{-\infty}^\infty \sqrt{5}e^{-(x-10)^2}dx, \int_{-\infty}^{10} \sqrt{5}e^{-(x-10)^2}dx$ and $\int_{10}^\infty \sqrt{5}e^{-(x-10)^2}dx$.

11. If $\alpha > 0$ constant, then prove $\int_0^\infty e^{-\alpha x^2}dx = \int_{-\infty}^0 e^{-\alpha x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$. Hence, for $\alpha > 0$ constant we have $\int_{-\infty}^\infty e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}}$.

12. Write the erf(x) and erfc (x) as power series, with $x \in \mathbb{R}$.

13. Let $f(x) = \int_0^x e^{-t^2}dt$. Prove $\int_0^\infty e^{-x^2+f(x)}dx = e^{\frac{\sqrt{\pi}}{2}} - 1$.

14. (a) Prove that for any real constant a

$$\int_0^\infty e^{\frac{-a}{x^2}}\,dx = \infty.$$

(b) If a and b are real constants, explain why we cannot split the integral

$$\int_0^\infty \left(e^{\frac{-a}{x^2}} - e^{\frac{-b}{x^2}}\right)dx.$$

as the difference

$$\int_0^\infty e^{\frac{-a}{x^2}}\,dx - \int_0^\infty e^{\frac{-b}{x^2}}\,dx.$$

(c) If $a \geq 0$ and $b \geq 0$ constants, then prove that

$$\int_0^\infty \left(e^{\frac{-a}{x^2}} - e^{\frac{-b}{x^2}}\right) dx = \sqrt{\pi b} - \sqrt{\pi a}.$$

[Hint: Use $u = \frac{1}{x}$, integration by parts, L' Hôpital's rule.

(d) If $a > b \geq 0$ constants, then prove that

$$\int_0^\infty \left(e^{\frac{a}{x^2}} - e^{\frac{b}{x^2}}\right) dx = \infty.$$

15. (a) Show that

$$\int_0^1 \ln(u)du = -1, \quad \int_0^1 |\ln(u)|du = 1 \quad \text{and}$$

$$\int_{-2}^2 \ln|u|du = 4[\ln(2) - 1].$$

(b) Show that $\forall m = 0, 1, 2, 3, \ldots$ integer

$$\int_0^1 \ln^m(u)du = (-1)^m m! \quad \text{and} \quad \int_0^1 |\ln(u)|^m du = m!.$$

Hint: Use an integral formula or induction. Remember $0! = 1$.]

16. (a) Use the known facts $0 < \sin(x) < x$ for $0 < x \leq \frac{\pi}{2}$, $\lim_{x\to 0^+} \frac{\sin(x)}{x} = 1^-$, and Problem 15 (a) to prove that

$$I = \int_0^{\pi/2} \ln[\sin(x)]dx$$

exists and has value < -1.

(b) Use the appropriate trigonometric properties of sine and cosine and u-substitutions to show that the integral I in (a) is also equal to

$$I = \int_{\pi/2}^{\pi} \ln[\sin(x)]dx = \int_0^{\pi/2} \ln[\cos(x)]dx = \int_{\pi/2}^{\pi} \ln[|\cos(x)|]dx$$

and

$$I = \int_0^1 \frac{\ln(u)}{\sqrt{1-u^2}}du = \frac{1}{2}\int_{-1}^1 \frac{\ln(|u|)}{\sqrt{1-u^2}}du.$$

(c) Prove the relation

$$2I = \int_0^{\pi/2} \ln\left[\frac{\sin(2x)}{2}\right] dx.$$

(d) Consider the relation in (c) and prove that the common value of the integrals in (a) and (b) is

$$I = -\frac{\pi}{2}\ln(2).$$

(e) Prove that $\forall a > 0$

$$\int_0^{\pi/2} \ln[a\sin(x)]dx = \int_0^{\pi/2} \ln[a\cos(x)]dx = \frac{\pi}{2}\ln\left(\frac{a}{2}\right).$$

(f) Prove

$$\int_0^{\pi} \ln[\sin(x)]dx = \int_0^{\pi} \ln[|\cos(x)|]dx = -\pi\ln(2).$$

17. Show

(a) $\displaystyle\int_0^{\pi/4} \ln[\sin(\theta)]d\theta = \int_{\pi/4}^{\pi/2} \ln[\cos(\theta)]d\theta$

(b) $\displaystyle\int_{\pi/4}^{\pi/2} \ln[\sin(\theta)]d\theta = \int_0^{\pi/4} \ln[\cos(\theta)]d\theta$

[Hint: Use Problem 16 for existence and let $u = \frac{\pi}{2} - \theta$.]

18. (a) Show
$$\int_0^{\pi/2} \ln[\tan(x)]dx.$$
exists and is equal to 0

(b) Then show
$$\int_0^{\pi} \ln[|\tan(x)|]dx = 0$$
[Hint: Use Problem 16 above.]

19. (a) Show that
$$\int_0^{\infty} \frac{\ln(x)}{x^2+1}dx$$
exists and equals 0

[Hint: Use $x = \tan(w)$ and Problems 18 and 15, (a), above.]

(b) Prove
$$\frac{\pi}{4} - \frac{\ln(2)}{2} < \int_1^{\infty} \frac{\ln(x)}{x^2+1}dx = \int_0^1 \frac{-\ln(x)}{x^2+1}dx < 1$$
[Hint: $\forall x > 0, 1 - \frac{1}{x} \leq \ln(x) \leq x - 1$. Prove this inequality!]

20. Prove
$$\int_0^1 \frac{\ln(x+1)}{x^2+1}dx = \frac{\pi\ln(2)}{8}$$
[Hint: Notice that this integral is proper. Use first $x = \tan(w)$ and then prove and use the relation $\sin(w)+\cos(w) = \sqrt{2}\sin\left(\frac{\pi}{4}+w\right) = \sqrt{2}\cos\left(\frac{\pi}{4}-w\right)$. You may also need Problem 17]

21. Prove
$$\int_{-1}^{\infty} \frac{\ln(x+1)}{x^2+1}dx = \frac{3\pi\ln(2)}{8}$$

[Hint: Look at the hint of Problem 20.]

22. Show that
$$\int_0^{\infty} \frac{\ln(x)}{(x^2+1)^2} dx = -\frac{\pi}{4}$$

[Hint: Use Problem 19 (a) and integration by parts.]

23. For any $r \in \mathbb{R}$ prove that
$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^r(x)} dx \overset{(x=\frac{\pi}{2}-u)}{=} \int_0^{\frac{\pi}{2}} \frac{\tan^r(u)}{1+\tan^r(u)} du^{\left[\tan(u)=\frac{1}{\cot(u)}\right]}$$
$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cot^r(x)} dx = \int_0^{\frac{\pi}{2}} \frac{\cot^r(u)}{1+\cot^r(u)} du = \frac{\pi}{4}.$$

24. If $\alpha > 0$ and $\beta > 0$, prove the two general formulae

   (a) $\int_0^{\infty} \frac{1}{\beta^2+x^2} dx = \frac{\pi}{2\beta}$ and

   (b) $\int_0^{\infty} \frac{\ln(\alpha x)}{\beta^2+x^2} dx = \frac{\pi}{2\beta} \ln(\alpha\beta)$.

   Hint: For the first integral, use arc-tangent. For the second integral, use Problem 19 and adjust.]

25. Consider the integral $\int_0^{\infty} e^{-x} \ln(x) dx$

   Prove:

   (a) This integral converges absolutely.

   (b) This integral is equal to
$$\int_0^1 \frac{e^{-t}+e^{\frac{-1}{t}}-1}{t} dt.$$

   (c) The integral in (b) is proper.

   (d) The value of these two equal integrals is negative.

5.5 Integrals Dependent on Parameters

For difficult integrals (proper or improper) that depend on parameters, we may use the Techniques of Continuity and Differentiability, as illustrated in this section. Again, let $A \subseteq \mathbb{R}$ be any typical set used in the definitions of the improper integrals, which we have examined in Chapter 5, Section 5.1.

We consider continuous or piecewise continuous functions $f(x,t)$ with $x \in A$ and t in some interval $I \subseteq \mathbb{R}$. If we consider the integral (proper or improper)

$$\int_A f(x,t)dx, \quad \forall\, t \in I,$$

then we call it an integral with a parameter, namely t.

We define the set

$$J = \left\{ t \mid t \in I \quad : \quad \int_A f(x,t)dx \text{ exists} \right\} \subseteq I$$

If this set is non-empty ($J \neq \emptyset$), this integral defines a function $F(t)$ on the set J and t is now viewed as a variable. Namely

$$F(t) = \int_A f(x,t)dx, \text{ with } t \in J$$

Depending on the definition of $f(x,t)$ and the set A, we have $\emptyset \subseteq J \subseteq I$. So, if $J = \emptyset$, there is nothing to talk about. Otherwise, in the Theorem that follows, we address the interesting case where $J = I \subseteq \mathbb{R}$ is an interval of the form (α, β) or $[\alpha, \beta]$ or $[\alpha, \beta)$ or $(\alpha, \beta]$, where $-\infty \leq \alpha < \beta \leq \infty$

From calculus, we know that when the integral is proper, hence $I = [a,b]$ is a closed and bounded interval with $-\infty < a < b < \infty$ constant real numbers, $f(x,t)$ is continuous and $\frac{\partial f(x,t)}{\partial t}$ is continuous, then $F(t)$ is differentiable and therefore continuous. But in case of an improper integral, even if $f(x,t)$ is continuous or differentiable, it does not follow automatically that $F(t)$ is continuous or differentiable at a given point $t_0 \in I$. To guarantee these outcomes, we need some extra conditions. Here we are going to state quite a general version of a Theorem for the continuity and differentiability of $F(t)$. Variations and generalizations of this Theorem may be found in advanced books, along with various proofs. Here we concentrate on the correct use of this Theorem as a tool for computing integrals, and we omit its proof as being above the level of this book. We like to refer to it as the Main Theorem of this section.

Theorem 5.5.1. *Let $f(x,t)$ be a real function "nice enough" in $x \in A \subseteq \mathbb{R}$ and continuous in $t \in I \subseteq \mathbb{R}$ where I is an interval of the form (α, β) or $[\alpha, \beta]$ or $[\alpha, \beta)$ or $(\alpha, \beta]$, with $-\infty < \alpha < \beta < \infty$*

(I) **Continuity** : *Suppose that there exists a real function $g(x) \geq 0$, nice in A, such that*

$$|f(x,t)| \leq g(x), \forall x \in A \text{ and } \forall t \in I, \qquad \text{and} \int_A g(x)dx < \infty.$$

Then the function

$$F(t) = \int_A f(x,t)dx, \ \text{with } t \in I,$$

is a continuous real valued function in I. (At an endpoint of I, the continuity is understood as the suitable left or right side continuity.) So, $F(t)$ satisfies

$$\forall t_0 \in I, \qquad \lim_{I \ni t \to t_0} F(t) = F(t_0) = F\left(\lim_{I \ni t \to t_0} t\right)$$

or

$$\lim_{I\ni t\to t_0}\int_A f(x,t)dx = \int_A f\left(x,t_0\right)dx = \int_A \lim_{I\ni t\to t_0} f(x,t)dx.$$

Under these conditions the same result is (obviously) true for the real valued function

$$G(t) := \int_A |f(x,t)|dx, \quad t\in I.$$

(II) **Differentiability** : *Suppose that* $\frac{\partial f(x,t)}{\partial t}$ *exists for* $t\in I$ *and* $x\in A$ *and suppose that there exists a real function* $g(x)\geq 0$ *nice in* A *such that*

$$\left|\frac{\partial f(x,t)}{\partial t}\right| \leq g(x), \forall x\in A \text{ and } \forall t\in I, \text{ and } \int_A g(x)dx < \infty$$

Then

$$F(t) = \int_A f(x,t)dx$$

is

$$\frac{dF}{dt}(t) = F'(t) = \int_A \frac{\partial f(x,t)}{\partial t}dx$$

Remark 5.5.1. On the previous theorem, we can make the following remarks.

1. The condition $\int_A g(x) < \infty$ implies the absolute convergence of $\int_A f(x,t)dx, \forall\ t\in I$
2. The power of this Theorem and use of parameter(s) in integrals are illustrated in several examples that follow.
3. To check continuity and differentiability of $F(t)$, we need to check either property point by point for any point t, where $\alpha < t <$

β. (Continuity and differentiability are local properties or pointwise properties.) So, we keep in mind that in order to do this, many times, we simply take any random $t \in (\alpha, \beta)$ and then a "small" interval $[c, d]$ or $(\alpha, d]$ or $[c, \beta)$ containing t and subset of (α, β). Then it is easier and more convenient to work over this new smaller subinterval of (α, β) for finding an appropriate choice of the function $g(x)$ over this subinterval only.

4. Sometimes we find a $g(x)$ for $f(x,t)\cdot$ (a damping factor), as we shall see in some examples that follow.

5. The first part of the Theorem is essentially due to Weierstraß. Both parts of this Theorem have been generalized in various ways by the Lebesgue theory of integration. This result is stronger than those which require the uniform convergence of an integral dependent on a parameter as we encounter in other expositions.

6. The second part of the Theorem proves the Leibniz rule for differentiation of Riemann integrals over bounded closed intervals. This states:

   If $f(x,t)$ is continuous in (x,t) and continuously differentiable in t, where $x \in [a,b] \subset \mathbb{R}$ and $t \in (\alpha, \beta) \subseteq \mathbb{R}$, then

   $$\frac{d}{dt}\int_a^b f(x,t)dx = \int_a^b \frac{\partial f(x,t)}{\partial t}dx, \quad \forall t \in (\alpha, \beta)$$

   Combining this rule with the chain rule (for the differentiation of composition of functions), we obtain the general Leibniz rule. This states:

   If $u(t)$ and $v(t)$ are differentiable real valued functions and $f(x,t)$

satisfies the above conditions in every interval $[u(t), v(t)]$ of x, then

$$\frac{d}{dt}\left[\int_{u(t)}^{v(t)} f(x,t)dx\right] =$$

$$\int_{u(t)}^{v(t)} \frac{\partial f(x,t)}{\partial t} dx + f[v(t),t]\cdot v'(t) - f[u(t),t]\cdot u'(t)$$

7. We can use the Leibniz Rule to evaluate new definite integrals from known ones that depend on parameters.

Examples using the Leibniz Rule

1. From the known integral

$$\int_0^b \frac{dx}{1+ax} = \frac{1}{a}\ln(1+ab)$$

with parameter $a > 0$ and upper limit $b > 0$ constant (check that this answer is correct), we obtain the following new integral formula

$$\int_0^b \frac{xdx}{(1+ax)^2} = \frac{1}{a^2}\ln(1+ab) - \frac{b}{a(1+ab)}$$

by differentiating both sides of the above equality with respect to the parameter a. (Compute the derivatives and confirm the correctness of the answer stated here.)

2. From the known integral

$$\int_0^b \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{b}{a}\right)$$

with parameter $a > 0$ and upper limit $b > 0$ constant (check that this answer is correct), we obtain the new integral formula

$$\int_0^b \frac{dx}{\left(a^2+x^2\right)^3} = \frac{b}{8a^4}\left[\frac{5a^2+3b^2}{\left(a^2+b^2\right)^2} + \frac{3}{ab}\arctan\left(\frac{b}{a}\right)\right]$$

by differentiating twice with respect to the parameter a and making the necessary adjustments. (Perform all missing steps and computations!)

3. Using calculus (the integration of rational expressions of sines and cosines), we find the integral formula

$$F(a,b) = \int_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2(x)+b^2\sin^2(x)} = \frac{\pi}{2ab}$$

with parameters $a>0$ and $b>0$. (We may achieve this result by the method of integrating rational functions of $\sin(x)$ and $\cos(x)$. Open a calculus book that contains this section and review this method one more time.)

Now, by differentiating with respect to the parameter a and multiplying by b and then differentiating with respect to the parameter b and multiplying by a, adding the results and dividing by $-2ab$, we obtain the following new integral formula

$$\frac{1}{-2ab}\left[b\frac{\partial F(a,b)}{\partial a} + a\frac{\partial F(a,b)}{\partial b}\right] =$$
$$\int_0^{\pi/2} \frac{dx}{\left[a^2\cos^2(x)+b^2\sin^2(x)\right]^2} = \frac{\pi\left(a^2+b^2\right)}{4(ab)^3}.$$

4. For your own practice, find known proper integrals with parameters from various sources, like tables, apply differentiation and necessary manipulation to derive new integral formulae, as we did

in the previous three examples.

5.6 The Real Gamma and Beta Functions

In this section, we study the real Gamma and Beta functions and their fundamental properties. These special functions are very important in Mathematics, Statistics, Engineering and Science. They were first defined and used by Euler.

The Gamma Function

The Gamma function is defined as the improper integral of a real parameter p

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx.$$

This is also called **Euler′s integral of the second kind**. We notice that the integrand is always non-negative, and this integral is obviously improper since the interval of integration, $[0, \infty)$, is unbounded. When $0 < p < 1$, the integral is improper for one more reason: The integrand becomes $+\infty$ at $x = 0^+$.(In complex analysis, the real p is replaced with the complex variable $z = x + iy$, and so we must rename the dummy variable with a letter other than x.)

In Example 5.3.8 (2) we have established the convergence of this integral for all $p > 0$ and its divergence $(= \infty)$ for all $p \leq 0$. So, the study of the Gamma function begins with examining the properties of this integral for $p \in (0, \infty)$.

We are also going to provide some preliminary estimates which, besides reproving the existence of $\Gamma(p)$ for all $0 < p < \infty$, allow us to see what happens when p approaches 0^+or ∞.

Preliminary Estimates

In proving various facts about the Gamma function, it is convenient to write

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx = \int_0^1 x^p \frac{1}{xe^x} dx + \int_1^\infty x^p \frac{1}{xe^x} dx \tag{5.2}$$

With the help of this relation, we will derive some preliminary estimates for the Gamma function. These estimates are useful in proving some results and solving some problems.

(I) We estimate the first part of the integral in (5.2), namely:

$$\int_0^1 x^{p-1} e^{-x} dx = \int_0^1 x^p \frac{1}{xe^x} dx \quad \text{for all} \quad p > 0$$

This integral is improper when $0 < p < 1$ and is proper when $p \geq 1$

We observe the following inequality:

$$\forall\ 0 < p < \infty \text{ and } \forall\ 0 \leq x \leq 1, \quad x^{p-1} e^{-1} \leq x^{p-1} e^{-x} \leq x^{p-1}.$$

Therefore,

$$\frac{1}{e} \int_0^1 x^{p-1} dx < \int_0^1 x^{p-1} e^{-x} dx < \int_0^1 x^{p-1} dx.$$

So

$$\forall\ 0 < p < \infty, \quad \frac{1}{ep} < \int_0^1 x^{p-1} e^{-x} dx < \frac{1}{p}.$$

(II) We now continue with the estimation of the second part of the integral in (5.2), namely:

$$\int_1^\infty x^{p-1}e^{-x}dx = \int_1^\infty x^p \frac{1}{xe^x}dx$$

This integral is improper because of the infinite interval of integration. We distinguish two cases:

case 1: $0 < p < 1$

By the following inequality (which deserves mentioning on its own merit; therefore, prove it as an exercise)

$$\forall\ x > 0, \quad 1 - \frac{1}{x} \leq \ln(x) \leq x - 1,$$

we see that for any $x \geq 1$ we have $0 \leq \ln(x) \leq x - 1$, and so we obtain: $\forall\ x \geq 1$ and $\forall\ 0 < p < 1$

$$e^{-2x+1} \leq e^{-x-\ln(x)} = x^{-1}e^{-x} \leq x^{p-1}e^{-x} \leq e^{-x}$$

Therefore, $\forall\ 0 < p < 1$,

$$\int_1^\infty e^{-2x+1}dx < \int_1^\infty x^{p-1}e^{-x}dx < \int_1^\infty e^{-x}dx$$

and so

$$\forall\ 0 < p < 1, \quad \frac{1}{2e} < \int_1^\infty x^{p-1}e^{-x}dx < \frac{1}{e}$$

case 2: $1 \leq p < \infty$

We let $n = [\![p-1]\!]$, the integer part of $p - 1$. Since $p \geq 1$, we have that $n \geq 0$ integer and $n \leq p - 1 < n + 1$. So, we have the inequality

$$\int_1^\infty x^n e^{-x} dx \leq \int_1^\infty x^{p-1} e^{-x} dx < \int_1^\infty x^{n+1} e^{-x} dx$$

Applying n integrations by parts in the first integral and $n+1$ integrations by parts in the third, we obtain
$\frac{1+n+n(n-1)+n(n-1)(n-2)+\ldots+n!+n!}{e}$
$\leq \int_1^\infty x^{p-1} e^{-x} dx$
$< \frac{1+(n+1)+(n+1)n+(n+1)n(n-1)+\ldots+(n+1)!+(n+1)!}{e}$

Finally, we have achieved the following **preliminary estimates of the Gamma function** :

(1)

$$\forall\ 0 < p < 1, \quad \frac{p+2}{2ep} = \frac{1}{2e} + \frac{1}{ep} < \Gamma(p) < \frac{1}{p} + \frac{1}{e} = \frac{e+p}{ep}$$

obtained by (I) and (II, Case **1**).

From these estimates, we also obtain the fact:

$$\lim_{p \to 0^+} \Gamma(p) = \infty$$

(2) $\forall\ 1 \leq p < \infty$

$$\frac{1}{ep} + \frac{1+n+n(n-1)+n(n-1)(n-2)+\ldots+n!+n!}{e}$$
$$< \Gamma(p) < \frac{1}{p} +$$
$$\frac{1+(n+1)+(n+1)n+(n+1)n(n-1)+\ldots+(n+1)!+(n+1)!}{e}$$

where $n = [\![p-1]\!]$, obtained by (I) and (II, Case **2**).

If $p \to \infty$, then $[\![p-1]\!] = n \to \infty$, and from these estimates we obtain the fact:

$$\lim_{p\to\infty} \Gamma(p) = \infty$$

Some Basic Properties and Values of the Gamma Function

$(\mathbf{\Gamma},\ \mathbf{1}):$ For $0 < p < \infty, \Gamma(p)$ is a continuous function of p.

Proof. We must use the Continuity Part of Theorem 5.5.1.

Since continuity is a local property, for any given
$0 < p < \infty$ we fix any p_1, p_2 such that $0 < p_1 < p < p_2 < \infty$.
Then we use the

Theorem by choosing the non-negative function

$$g(x) = \begin{cases} x^{p_1-1}e^{-x}, & \text{if } 0 < x \leq 1 \\ x^{p_2-1}e^{-x}, & \text{if } 1 \leq x < \infty. \end{cases}$$

Then

$$\left|x^{p-1}e^{-x}\right| = x^{p-1}e^{-x} < g(x)$$

and

$$\int_0^\infty g(x)dx < \Gamma(p_1) + \Gamma(p_2) < \infty.$$

Now the result follows from the Continuity Part of Theorem 5.5.1. □

$(\mathbf{\Gamma},\ \mathbf{2}):$ For $0 < p < \infty,\ \Gamma(p)$ is infinitely differentiable with $\mathbf{n}^{\text{th}}$ order derivative

$$\Gamma^{(n)}(p) = \int_0^\infty x^{p-1}e^{-x}[\ln(x)]^n dx, \forall n = 0, 1, 2, 3, \ldots$$

Proof. For the first derivative and similarly with any derivative thereon, we use the Differentiability Part of Theorem 5.5.1. Since differentiability is a local property, for any given $0 < p < \infty$ we fix any p_1, p_2 such that $0 < p_1 < p < p_2 < \infty$. We deal with the function

$$\left| \frac{\partial}{\partial p} \left(x^{p-1}e^{-x} \right) \right| = x^{p-1}e^{-x} |\ln(x)|.$$

Then we use the Theorem by choosing the non-negative function

$$g(x) = \begin{cases} x^{p_2}e^{-x}, & \text{if } 1 \leq x < \infty \\ x^{p_1-1}[-\ln(x)], & \text{if } 0 < x < 1. \end{cases}$$

We see that

$$\left| \frac{\partial}{\partial p} \left(x^{p-1}e^{-x} \right) \right| = x^{p-1}e^{-x} |\ln(x)| < g(x)$$

and we also observe

$$\int_1^\infty g(x)dx = \int_1^\infty x^{p_2}e^{-x}dx < \Gamma(p_2 + 1) < \infty.$$

For the other part of the integral of $g(x)$, we use u-substitution and integration by parts to obtain

$$\int_0^1 x^{p_1-1}[-\ln(x)]dx = \int_0^1 x^{p_1-1}\ln\left(\frac{1}{x}\right)dx \overset{u=\frac{1}{x}}{=}$$
$$\int_\infty^1 u^{1-p_1}\ln(u)\frac{du}{-u^2} = \int_1^\infty u^{-1-p_1}\ln(u)du = \frac{1}{p_1^2} < \infty.$$

Hence

$$\int_0^\infty g(x)dx < \infty$$

and the result follows.

Note :

$$\Gamma'(1) = \int_0^\infty e^{-x}\ln(x)dx = -\gamma < 0,$$

where γ is the Euler constant defined to be

$$\gamma = \lim_{n\to\infty}\left[\left(\sum_{k=1}^n \frac{1}{k}\right) - \ln(n)\right] \simeq 0.57721566\ldots > 0.$$

Make a note of this result, for it is needed in the proofs of many important and difficult results on special integrals and special functions. □

($\boldsymbol{\Gamma}$, 3) : The Gamma function is analytic. That is, it can be expressed as a power series locally.

[Hint: Use the Taylor Power Series Theorem as we know it from calculus and then show that for any fixed point $0 < p_0 < \infty$ the Taylor remainder

$$R_n = \frac{\Gamma^{(n+1)}(p^*)}{(n+1)!}(p-p_0)^{n+1}$$

with some p^* between p and p_0, approaches zero as $n \to \infty$.

Then show that the radius of convergence of the obtained power series is $R = p_0 \cdot$]

$(\mathbf{\Gamma},\ \mathbf{4}):$ For $0 < p < \infty, \Gamma(p)$ is strictly convex. This follows by $(\Gamma,\ 2)$ since

$$\Gamma''(p) = \int_0^\infty x^{p-1} e^{-x} [\ln(x)]^2 dx > 0.$$

Hence the second derivative of the Gamma function is strictly positive, and therefore the Gamma function is strictly convex (or concave up, as we say in calculus).

$(\mathbf{\Gamma},\ \mathbf{5}):$ $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof.

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx \overset{x=u^2}{=}$$
$$2\int_0^\infty e^{-u^2} du = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

by the integral (5.1) in section 5.4. □

$(\mathbf{\Gamma},\ \mathbf{6}):$ $\Gamma(1) = 1$ and $\Gamma(2) = 1$

Proof.

$$\Gamma(1) = \int_0^\infty e^{-x} dx = \left[-e^{-x}\right]_0^\infty = 0 - (-1) = 1$$
$$\Gamma(2) = \int_0^\infty x e^{-x} dx = \left[-x e^{-x}\right]_0^\infty + \int_0^\infty e^{-x} dx = 0 + 1 = 1$$

□

(Γ, 7) : By $(\Gamma, 4)$ and $(\Gamma, 6)$, we conclude that the Gamma function has a unique local minimum, which is also global minimum in $(0, \infty)$, at a number r between 1 and 2. An estimate of this number is

$$r = 1.46163214496836234126265954232572132846819 6\ldots$$

and the global minimum of the Gamma function in $(0, \infty)$ is approximately

$\Gamma(r) = 0.88560319441088870027881590058258873320795\ldots$
We then conclude that $\Gamma'(r) = 0, \Gamma(p)$ is strictly decreasing [and so $\Gamma'(p) < 0$] for $0 < p < r$ and strictly increasing [and so $\Gamma'(p) > 0$] for $r < p < \infty$

(Γ, 8) : $\Gamma(p+1) = p\Gamma(p)$ for $p > 0$

Proof.

$$\begin{aligned}
\Gamma(p+1) &= \int_0^\infty x^p e^{-x} dx \\
&= \text{ (use integration by parts)} \\
&= \left[-x^p e^{-x}\right]_0^\infty + \int_0^\infty p x^{p-1} e^{-x} dx \\
&= 0 + p \int_0^\infty x^{p-1} e^{-x} dx \\
&= p\Gamma(p).
\end{aligned}$$

We can use this recursive relation as

$$\Gamma(p) = \frac{1}{p}\Gamma(p+1)$$

to extend the Gamma function to the negative non-integer real numbers (recursively).

For instance, if $p = -\frac{2}{3}$, we find

$$\Gamma\left(-\frac{2}{3}\right) = -\frac{3}{2}\Gamma\left(\frac{1}{3}\right)$$

In the same way, keeping in mind that $\Gamma\left(0^{+}\right) = \infty$ and $\Gamma(1) = 1$, we get that the real Gamma function approaches $\pm\infty$ as p approaches a negative integer. More precisely

$$\text{for } n \leq 0 \text{ integer, } \Gamma\left(n^{\pm}\right) = (-1)^{n}(\pm\infty)$$

For instance: $\Gamma\left(0^{-}\right) = -\infty, \Gamma\left(-1^{+}\right) = -\infty, \Gamma\left(-1^{-}\right) = +\infty$, etc (See Figure 5.5). $\square$

$(\boldsymbol{\Gamma},\ \mathbf{9})$: For $p = 0, 1, 2, 3, \ldots, \Gamma(p+1) = p!$

Proof. Do p iterations of $(\Gamma, 8)$ and use $(\Gamma, 6)$ $\square$

We now notice that the Gamma function is an analytic function in $(0, \infty)$ and contains all the factorials $n!$ of the integers $n = 0, 1, 2, 3, \ldots$ in the range of its values. Therefore, the Gamma function is used to define the factorials of all real numbers greater than -1. In fact, we define:

$$\forall\ p > -1, \quad p! = \Gamma(p+1).$$

For example:

$$\Gamma(20) = 19!$$

and

$$0! = \Gamma(1) = 1, \quad 1! = \Gamma(2) = 1, \quad \text{and so on}$$

Since by $(\Gamma, 8)$ and $(\Gamma, 5)$, we obtain

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

we write $\left(\frac{1}{2}\right)! = \frac{1}{2}\sqrt{\pi}$.

It is clear now that if $p > 0$ is not an integer, then we can evaluate $\Gamma(p)$ by means of the value of $\Gamma(p-p)$, where p is the integer part of p. That is, all values $\Gamma(p)$ may be found in terms of the factorials of integers and the values of $\Gamma(p)$ for $0 < p < 1$ (Years ago, there were in use extensive tables of values of $\Gamma(p)$ for many $0 < p < 1$.)

(Γ, 10) :

$$\Gamma\left(p+\frac{1}{2}\right) = \Gamma\left(p-\frac{1}{2}+1\right) = \left(p-\frac{1}{2}\right)\Gamma\left(p-\frac{1}{2}\right) = \left(\frac{2p-1}{2}\right)\Gamma\left(p-\frac{1}{2}\right) = \ldots \text{ etc.}$$

(Γ, 11) : For $x > 0$ and $p > 0$, by making the substitution $v = xu$, we obtain

$$\frac{1}{x^p} = \frac{1}{\Gamma(p)}\int_0^\infty u^{p-1}e^{-xu}du, \quad \text{or} \quad \int_0^\infty u^{p-1}e^{-xu}du = \frac{\Gamma(p)}{x^p}$$

If $p = 1, 2, 3, \ldots$ integer, then

$$\int_0^\infty u^{p-1}e^{-xu}du = \frac{(p-1)!}{x^p}.$$

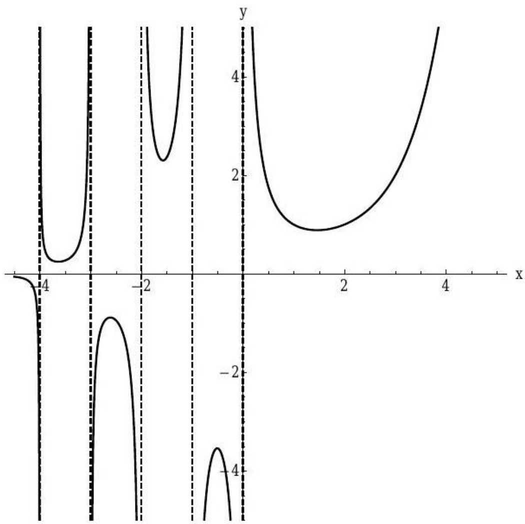

Figure 5.5: Graph of the Gamma function and its vertical asymptotes at $x = n$, where $n - 0, -1, -2, -3 \cdots$.

Result 5.6.1. With the help of the Gamma function, we can prove the renowned Stirling's formula which is useful for the computation of the factorials of large natural numbers. This says that if n is a large natural number, then we have the following approximate equality:

$$n! \cong \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Note 5.6.2. We now need some convenient conditions that guarantee the equality:

$$\iint\limits_{\mathbb{R}^2} f(x,y)dxdy =$$

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)dx\right]dy = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)dy\right]dx.$$

Following are some convenient conditions which guarantee the validity of equality.

Condition I.

$$f(x, y) \geq 0, \ \forall \ (x, y) \in \mathbb{R}^2,$$

or

$$f(x, y) \leq 0, \ \forall \ (x, y) \in \mathbb{R}^2.$$

Notice that in this case, the three parts of above equality may be all ∞ or $-\infty$, respectively. So, if the function is non-negative or non-positive (i.e., does not change sign) we can freely switch the order of integration in any way we would like without altering the answer.
Otherwise, we may use any one of the following three convenient conditions, which in real analysis are proven to be equivalent. I.e., any one of them implies the other two.

Condition II.

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |f(x, y)| dx \right] dy < \infty.$$

Condition III.

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |f(x, y)| dy \right] dx < \infty.$$

Condition IV.

$$\iint\limits_{\mathbb{R}^2} |f(x, y)| dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)| dx dy < \infty.$$

Here, the double integral is the limit of the double Riemann Sums of $|f(x, y)|$, as the norms of the double partitions approach 0. The equivalent conditions II and III were proved by **Tonelli**. The IVth equivalent condition was proved by **Fubini**.

Example 5.6.3.

1. $\int_0^\infty x^7 e^{-x} dx = \Gamma(8) = 7! = 5040.$

2.

$$\begin{aligned}\int_0^\infty x^4 e^{-2x} dx &\overset{u=2x}{=} \int_0^\infty \frac{u^4}{16} \cdot e^{-u} \frac{du}{2} \\ &= \frac{1}{32} \int_0^\infty u^4 e^{-u} du \\ &= \frac{1}{32} \cdot \Gamma(5) \\ &= \frac{1}{32} \cdot 4! \\ &= \frac{24}{32} = \frac{3}{4}.\end{aligned}$$

Or, by $(\Gamma, 11)$, we directly find

$$\int_0^\infty x^4 e^{-2x} dx = \int_0^\infty x^{5-1} e^{-2x} dx = \frac{\Gamma(5)}{2^5} = \frac{4!}{32} = \frac{24}{32} = \frac{3}{4}.$$

3. By $(\Gamma, 11)$, e.g., we find:

(a) $\displaystyle\int_0^\infty x^{4.57} e^{-3.5x} dx = \int_0^\infty x^{5.57-1} e^{-3.5x} dx = \frac{\Gamma(5.57)}{(3.5)^{5.57}},$

(b)

$$\begin{aligned}\int_0^\infty x^{0.5}e^{-0.2x}dx &= \int_0^\infty x^{1.5-1}e^{-0.2x}dx\\ &= \frac{\Gamma(1.5)}{(0.2)^{1.5}}\\ &= \frac{0.5\sqrt{\pi}}{0.2\sqrt{0.2}}\\ &= 2.5\sqrt{5\pi} = 9.908318244\ldots\end{aligned}$$

4.

$$\begin{aligned}\Gamma(4.01) &= \Gamma(3.01+1)\\ &= 3.01\cdot\Gamma(3.01)\\ &= 3.01\cdot 2.01\cdot 1.01\cdot 0.01\cdot\Gamma(0.01)\\ &= 0.06110601\cdot\Gamma(0.01)\end{aligned}$$

5.

$$\begin{aligned}\int_0^\infty \sqrt{y}e^{-y^2}dy &\overset{x=y^2}{=} \int_0^\infty x^{\frac{1}{4}}e^{-x}\frac{dx}{2\sqrt{x}}\\ &= \frac{1}{2}\int_0^\infty x^{\frac{-1}{4}}e^{-x}dx\\ &= \frac{1}{2}\int_0^\infty x^{\frac{3}{4}-1}e^{-x}dx\\ &= \frac{1}{2}\cdot\Gamma\left(\frac{3}{4}\right).\end{aligned}$$

6.

$$\begin{aligned}
\int_0^\infty \sqrt[3]{y} e^{-y^2} dy &\overset{x=y^2}{=} \int_0^\infty x^{\frac{1}{6}} e^{-x} \frac{dx}{2\sqrt{x}} \\
&= \frac{1}{2} \int_0^\infty x^{\frac{-1}{3}} e^{-x} dx \\
&= \frac{1}{2} \int_0^\infty x^{\frac{2}{3}-1} e^{-x} dx \\
&= \frac{1}{2} \cdot \Gamma\left(\frac{2}{3}\right).
\end{aligned}$$

7. We can use $(\Gamma, 11)$ and double integration to show the following two useful results:

   (a) For $0 < p < 2$, $\int_0^\infty \frac{\sin(x)}{x^p} dx = \frac{\pi}{2\Gamma(p)\sin\left(\frac{p\pi}{2}\right)}$.

   (b) For $0 < p < 1$, $\int_0^\infty \frac{\cos(x)}{x^p} dx = \frac{\pi}{2\Gamma(p)\cos\left(\frac{p\pi}{2}\right)}$.

Both of these generalized Riemann integrals exist and are continuous in p within the respective intervals of p. We have stated that for $0 < p < 1$, the integrals are conditionally convergent. (They do not exist as Lebesgue integrals, but this is outside the scope of this text!) We have also seen that the first integral is conditionally convergent for $p = 1$ and absolutely convergent for $1 < p < 2$.

For all the other positive values of p, they are equal to infinity because of the singularity near $x = 0$. For $p \leq 0$, they do not exist because they oscillate "badly" "near" infinity.

Here, we prove the first one, and in analogous way we prove the second one. In $(\Gamma, 11)$, we have

$$\forall x > 0, \quad \forall p > 0, \quad \frac{1}{x^p} = \frac{1}{\Gamma(p)} \int_0^\infty u^{p-1} e^{-xu} du$$

and so

$$\begin{aligned}\int_0^\infty \frac{\sin(x)}{x^p}dx &= \lim_{M\to\infty}\int_0^M \frac{1}{\Gamma(p)}\int_0^\infty \left(u^{p-1}e^{-xu}du\right)\sin(x)dx\\ &= \frac{1}{\Gamma(p)}\lim_{M\to\infty}\int_0^M \left(\int_0^\infty u^{p-1}e^{-xu}du\right)x\frac{\sin(x)}{x}dx.\end{aligned}$$

Since

$$\forall x\in\mathbb{R},\quad \left|\frac{\sin(x)}{x}\right|\le 1,$$

for the positive function $h(x,u)=u^{p-1}e^{-xu}x$ in $(0,M]\times(0,\infty)$, we get

$$\begin{aligned}\int_0^M \frac{1}{\Gamma(p)}\left(\int_0^\infty u^{p-1}e^{-xu}du\right)xdx &= \int_0^M \frac{1}{x^p}xdx\\ &= \left[\frac{x^{2-p}}{2-p}\right]_0^M\\ &= \frac{M^{2-p}}{2-p}\\ &< \infty,\quad \forall\ 0<p<2.\end{aligned}$$

Hence, by the Tonelli conditions, we can switch the order of integration and obtain

$$\begin{aligned}
&\int_0^\infty \frac{\sin(x)}{x^p}dx \\
&= \frac{1}{\Gamma(p)} \lim_{M\to\infty} \int_0^\infty u^{p-1} \left(\int_0^M e^{-xu} \sin(x)dx \right) du \\
&= \frac{1}{\Gamma(p)} \lim_{M\to\infty} \int_0^\infty u^{p-1} \left[\frac{-e^{-xu}[u\sin(x)+\cos(x)]}{u^2+1} \right]_0^M du \\
&= \frac{1}{\Gamma(p)} \lim_{M\to\infty} \int_0^\infty u^{p-1} \frac{-e^{-Mu}[u\sin(M)+\cos(M)]+1}{u^2+1} du \\
&= \frac{1}{\Gamma(p)} \lim_{M\to\infty} \int_0^\infty \left[\frac{u^{p-1}}{u^2+1} - \frac{u^{p-1}e^{-Mu}[u\sin(M)+\cos(M)]}{u^2+1} \right] du.
\end{aligned}$$

Now, independently of the limit, we have

$$\forall\ 0 < p < 2, \qquad \int_0^\infty \frac{u^{p-1}}{u^2+1}du = \frac{1}{2}\frac{\pi}{\sin\left(\frac{p\pi}{2}\right)}.$$

Also, $\forall\ u > 0$ we have, $e^{-Mu} \to 0$ as $M \to \infty$. Thus,

$$\forall u > 0, \quad \forall\ 0 < p < 2, \quad \lim_{M\to\infty} \frac{u^{p-1}e^{-Mu}[u\sin(M)+\cos(M)]}{u^2+1} \overset{pw}{=} 0$$

Since $u > 0$ and $M \to \infty$, for $M \geq 1$, we have

$$\begin{aligned}
\left| \frac{u^{p-1}e^{-Mu}[u\sin(M)+\cos(M)]}{u^2+1} \right| &< \frac{u^{p-1}e^{-u}(u+1)}{u^2+1} \\
&< u^p e^{-u} + \frac{u^{p-1}}{u^2+1}.
\end{aligned}$$

Therefore, for any p such that $0 < p < 2$, we consider the positive function $g(u) = u^p e^{-u} + \dfrac{u^{p-1}}{u^2+1} > 0$ on $(0, \infty)$. This function is independent of M and satisfies

$$\begin{aligned}\int_0^\infty g(u)du &= \int_0^\infty \left(u^p e^{-u} + \frac{u^{p-1}}{u^2+1}\right) du \\ &= \int_0^\infty u^p e^{-u} du + \int_0^\infty \frac{u^{p-1}}{u^2+1} du \\ &= \Gamma(p+1) + \frac{1}{2}\frac{\pi}{\sin\left(\frac{p\pi}{2}\right)} < \infty,\end{aligned}$$

by the definition of the $\Gamma(p+1)$.

So, for any given p, such that $0 < p < 2$, by Part (I) of the Main Theorem, we obtain

$$\lim_{M\to\infty} \int_0^\infty \frac{u^{p-1}e^{-Mu}[u\sin(M)+\cos(M)]}{u^2+1} du =$$
$$\int_0^\infty \lim_{M\to\infty} \frac{u^{p-1}e^{-Mu}[u\sin(M)+\cos(M)]}{u^2+1} du = \int_0^\infty 0 du = 0.$$

This limit, along with the result in Examples and relation above proves the first equality.

Notice that both formulae this example are discontinuous at $p = 0$ I.e., at $p = 0$ the two sides of the formulae do not agree.

Example 5.6.4. The following two more general results are very useful:

1. for α and $\beta \neq 0$ real constants such that $-1 < \dfrac{\alpha+1}{\beta} < 1$, we have

$$\int_0^\infty x^\alpha \sin\left(x^\beta\right) dx = \frac{1}{|\beta|} \cdot \frac{\pi}{2\cdot\Gamma\left(1-\frac{\alpha+1}{\beta}\right)\cdot\sin\left[\left(1-\frac{\alpha+1}{\beta}\right)\frac{\pi}{2}\right]},$$

   and

2. for α and $\beta \neq 0$ real constants such that $0 < \frac{\alpha+1}{\beta} < 1$, we have

$$\int_0^\infty x^\alpha \cos\left(x^\beta\right) dx = \frac{1}{|\beta|} \cdot \frac{\pi}{2 \cdot \Gamma\left(1 - \frac{\alpha+1}{\beta}\right) \cdot \cos\left[\left(1 - \frac{\alpha+1}{\beta}\right)\frac{\pi}{2}\right]}.$$

The Beta Function

The Beta function is defined as the following integral

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx$$

with two real parameters p and q. This is also called Euler's integral of the first kind. We will shortly see that the Beta function is closely related to the Gamma function. (In complex analysis, p and q are considered complex variables.)

Basic Properties of the Beta Function

$(B,1)$: This integral converges to a positive finite value and it is continuous for $p > 0$ and $q > 0$. It diverges otherwise, i.e., it becomes infinite.

This result is obtained by proving the following four cases:

(a) For $p \geq 1$ and $q \geq 1$, the integral is a proper integral of a continuous function on $[0,1]$. So, its value is positive finite, in this case.

(b) For $p \leq 0$ or $q \leq 0$, we have $B(p,q) = \infty$.

(c) If $0 < p < 1$ and $0 < q < 1$, then we split the integral about $x = \frac{1}{2}$ to easily obtain convergence to a positive finite value.

(d) For $0 < p < 1$ and $q \geq 1$, or $p \geq 1$ and $0 < q < 1$, we also obtain convergence to a positive finite value easily.

$(B, 2)$: $\forall p > 0, \quad \forall q > 0, \quad B(p, q) = B(q, p)$

This is obtained by the change of variables $u = 1 - x$

$(B, 3)$: The Beta function satisfies

$$\forall p > 0, \quad \forall q > 1, \quad B(p, q) = \frac{q-1}{p} B(p, q-1) - \frac{q-1}{p} B(p, q).$$

[Hint: This relation is obtained by first using integration with parts, and then we replace x^p with $x^{p-1} - x^{p-1}(1-x)$.]

Similarly,

$$\forall p > 1, \quad \forall q > 0, \quad B(p, q) = \frac{p-1}{q} B(p-1, q) - \frac{p-1}{q} B(p, q).$$

Therefore, solving these two relations for $B(p, q)$, we obtained two recursive formulae of the Beta function

$$\forall p > 0, \quad \forall q > 1, \quad B(p, q) = \frac{q-1}{p+q-1} B(p, q-1)$$

and

$$\forall p > 1, \quad \forall q > 0, \quad B(p, q) = \frac{p-1}{p+q-1} B(p-1, q)$$

$(B, 4)$: By using the previous property, or directly, we can prove two more recursive formulae of the Beta function

$$\forall p > 0, \quad \forall q > 1, \quad B(p, q) = \frac{q-1}{p} B(p+1, q-1),$$

and

$$\forall p > 1, \quad \forall q > 0, \quad B(p, q) = \frac{p-1}{q} B(p-1, q+1).$$

$(B, 5)$: We have the following five integral representations of the Beta function: $\forall p > 0$ and $\forall q > 0$

$$\text{(I)} \quad B(p, q) = \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} du.$$

This is obtained by making the change of variables $x = \frac{u}{1+u}$.

By the symmetry of the Beta function, or by analogous substitution, we also have

$$(II) \quad B(p, q) = \int_0^\infty \frac{u^{q-1}}{(1+u)^{p+q}} du.$$

If we add equations (I) and (II) and divide by 2, we get

$$(III) \quad B(p, q) = \frac{1}{2} \int_0^\infty \frac{u^{p-1} + u^{q-1}}{(1+u)^{p+q}} du.$$

We observe that by making the substitution $u = \dfrac{1}{v}$ in the first integral below, we obtain

$$\int_0^1 \frac{u^{p-1} + u^{q-1}}{(1+u)^{p+q}} du = \int_1^\infty \frac{u^{p-1} + u^{q-1}}{(1+u)^{p+q}} du.$$

Therefore, by (III), we get

$$(IV) \quad B(p, q) = \int_0^1 \frac{u^{p-1} + u^{q-1}}{(1+u)^{p+q}} du,$$

and

$$(V) \quad B(p, q) = \int_1^\infty \frac{u^{p-1} + u^{q-1}}{(1+u)^{p+q}} du.$$

$(B, 6)$: $\forall p > 0, \forall q > 0, \quad B(p, q) = 2 \displaystyle\int_0^{\pi/2} \sin^{2p-1}(\theta) \cos^{2q-1}(\theta) d\theta$ This is obtained by letting $0 \le x = \sin^2(\theta) \le 1$

$(B, 7)$: Relation of the Beta function to the Gamma function:

$$\forall p > 0, \quad \forall q > 0, \quad B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

This follows by letting $x = u^2$ in $\Gamma(p)$ and $x = v^2$ in $\Gamma(q)$ to get

$$\begin{aligned}\Gamma(p)\Gamma(q) &= 4\left(\int_0^\infty u^{2p-1}e^{-u^2}du\right)\left(\int_0^\infty v^{2q-1}e^{-v^2}dv\right)\\ &= 4\int_0^\infty\int_0^\infty u^{2p-1}v^{2q-1}e^{-\left(u^2+v^2\right)}dudv.\end{aligned}$$

Now we use polar coordinates $u = r\cos\theta, v = r\sin\theta$ and $(B, 6)$ to find $\Gamma(p)\Gamma(q) = \Gamma(p+q)B(p,q)$, and the result follows.

From this result and $(\Gamma, 9)$, we get the convenient byproduct: For $m \ge 0$ and $n \ge 0$ integers, we have

$$\begin{aligned}\int_0^1 x^m(1-x)^n dx &= \int_0^1 x^n(1-x)^m dx\\ &= B(m+1, n+1) = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}\\ &= \frac{m!n!}{(m+n+1)!}\\ &= \frac{1}{(m+n+1)\begin{pmatrix} m+n \\ m \end{pmatrix}}\\ &= \frac{1}{(m+n+1)\binom{m+n}{n}}.\end{aligned}$$

$(B, 8)$: The Beta and Gamma functions satisfy:

$$\forall p: \quad 0 < p < 1, \quad B(p, 1-p) = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}.$$

To prove this, we let $q = 1 - p$ ($\Longleftrightarrow p + q = 1$) in $(B, 5, I)$ and use the result: $\displaystyle\int_0^{\infty} \frac{t^{p-1}}{1+t} dt = \frac{\pi}{\sin(p\pi)}$ to find first

$$\forall\ 0 < p < 1, \quad B(p, 1-p) = \int_0^{\infty} \frac{u^{p-1}}{1+u} du = \frac{\pi}{\sin(p\pi)}.$$

Since $\Gamma(p+q) = \Gamma(1) = 1$, by $(B, 7)$, we get that

$$\forall\ p : 0 < p < 1, \quad B(p, 1-p) = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}.$$

We can rewrite the last relation in various ways. For example, replacing p with $\frac{1}{2} + p$, we obtain:

$$\forall p : -\frac{1}{2} < p < \frac{1}{2}$$

$$B\left(\frac{1}{2} + p, \frac{1}{2} - p\right) = \Gamma\left(\frac{1}{2} + p\right)\Gamma\left(\frac{1}{2} - p\right) = \frac{\pi}{\cos(p\pi)}.$$

and so on.

$(B, 9)$: $\forall\ p > 0, \quad \forall\ q > 0, \quad \forall r > 0$ constants, we have

$$\int_0^1 \frac{u^{p-1}(1-u)^{q-1}}{(r+u)^{p+q}} du = \frac{1}{(r+1)^p \cdot r^q} B(p, q).$$

To obtain this result, use the change of variables $u = \frac{rx}{r+1-x}$

So, we have obtained the following expression of the Beta function:

$$\forall p > 0, \quad \forall q > 0, \quad \forall r > 0$$

$$B(p,q) = (r+1)^p \cdot r^q \int_0^1 \frac{u^{p-1}(1-u)^{q-1}}{(r+u)^{p+q}} du$$

$(B, 10)$: We can use the Beta function to obtain the so-called Gamma function duplication formula.

We have the relation

$$\begin{aligned} 2^{2p} \int_0^{\pi/2} \sin^{2p}(x)\cos^{2p}(x)dx &= \int_0^{\pi/2} \sin^{2p}(2x)dx \\ &\overset{u=2x}{=} \frac{1}{2}\int_0^{\pi} \sin^{2p}(u)du \\ &= \int_0^{\pi/2} \sin^{2p}(u)du. \end{aligned}$$

Then by $(B, 6)$, after ignoring the $1/2$ in both sides, we find

$$2^{2p} \cdot B\left(p+\frac{1}{2}, p+\frac{1}{2}\right) = B\left(p+\frac{1}{2}, \frac{1}{2}\right)$$

So, by $(B, 7)$

$$2^{2p} \cdot \frac{\Gamma^2\left(p+\frac{1}{2}\right)}{\Gamma(2p+1)} = \frac{\Gamma\left(p+\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma(p+1)}$$

By$(\Gamma, 8), \Gamma(x+1) = x\Gamma(x)$ for any $x > 0$, and $(\Gamma, 5), \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we find

$$2^{2p} \cdot \frac{\Gamma^2\left(p+\frac{1}{2}\right)}{2p\Gamma(2p)} = \frac{\Gamma\left(p+\frac{1}{2}\right)\cdot\sqrt{\pi}}{p\Gamma(p)}$$

Simplifying this and solving for $\Gamma(2p)$ we get:

$$\Gamma(2p) = \frac{2^{2p-1}}{\sqrt{\pi}} \cdot \Gamma(p) \cdot \Gamma\left(p + \frac{1}{2}\right) = \frac{2^{2p-\frac{1}{2}}}{\sqrt{2\pi}} \cdot \Gamma(p) \cdot \Gamma\left(p + \frac{1}{2}\right)$$

This formula, for obvious reasons, is called the Gamma function duplication formula.

From this formula, we obtain the useful formula

$$\Gamma\left(p + \frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2p)}{2^{2p-1}\Gamma(p)}$$

By this and $(\Gamma, 5)$, we also find that

$$\lim_{p\to 0^+} \frac{\Gamma(2p)}{\Gamma(p)} = \frac{\Gamma\left(\frac{1}{2}\right)}{2\sqrt{\pi}} = \frac{\sqrt{\pi}}{2\sqrt{\pi}} = \frac{1}{2}$$

$(B, 11)$: For any $p > 0$ fixed, we apply Lebasgue's Dominated Convergence Theorem with dominating function $g(x) = (1-x)^{p-1} \geq 0$, for $0 \leq x \leq 1$, to find

$$\lim_{0<q\to\infty} B(p, q) = 0^+$$

Therefore, by $(B, 7)$,

$$\lim_{0<q\to\infty} \frac{\Gamma(p+q)}{\Gamma(q)} = \lim_{0<q\to\infty} \frac{\Gamma(p)}{B(p, q)} = \frac{\Gamma(p)}{0^+} = +\infty$$

$(B, 12)$: For any $p > 0$ fixed, we have $\lim_{0<q\to\infty} q^p \cdot B(p, q) = \Gamma(p)$.

To prove this, we notice that for any $p > 0$ and any $q > 0$, we have

$$\begin{aligned} q^p \cdot B(p,q) &= \int_0^1 (qx)^{p-1}(1-x)^{q-1} d(qx) \\ &\overset{u=qx}{=} \int_0^q u^{p-1}\left(1-\frac{u}{q}\right)^{q-1} du \end{aligned}$$

But $\left(1-\frac{u}{q}\right)^{q-1} \uparrow e^{-u}$, as $q \longrightarrow \infty$. So by Lebasgue's Monotone Convergence Theorem, we obtain

$$\lim_{0<q\to\infty} q^p \cdot B(p,q) = \int_0^\infty u^{p-1}e^{-u} du = \Gamma(p).$$

This result implies the following important result for the Gamma function: For any $p \in \mathbb{R}$ fixed, we have

$$\lim_{0<q\to\infty} \frac{\Gamma(p+q)}{\Gamma(q)\cdot q^p} = 1.$$

For $p > 0$, we use $(B, 7)$ to get

$$\lim_{0<q\to\infty} \frac{\Gamma(p)}{q^p \cdot B(p,q)} = \lim_{0<q\to\infty} \frac{\Gamma(p+q)}{\Gamma(q)\cdot q^p} = 1.$$

For $p = 0$, we observe this result is trivially true.

For $p = -r < 0$ (and so $r > 0$), we manipulate the case with p positive to get that for given $r > 0$, we have

$$\lim_{0<q\to\infty} \frac{\Gamma(p-r)}{\Gamma(q)\cdot q^{-r}} = \lim_{0<q\to\infty} \frac{\left[\frac{(q-r)+r}{q-r}\right]^r}{\frac{\Gamma[(q-r)+r]}{(q-r)^r\cdot\Gamma(q-r)}} = \frac{1}{1} = 1.$$

Example 5.6.5.

A very large number of improper and/or complicated integrals are reduced to values of the Gamma and Beta functions. Many books on this subject contain a great number of them. Here we present the following:

1.

$$\begin{aligned}\int_0^1 x^{17}dx &= \int_0^1 (1-x)^{17}dx \\ &= B(18,1) = B(1,18) \\ &= \frac{\Gamma(18)\Gamma(1)}{\Gamma(19)} \\ &= \frac{17!\cdot 0!}{18!} \\ &= \frac{1}{18}.\end{aligned}$$

2.

$$\begin{aligned}\int_0^1 x^{17}(1-x)^{33}dx &= \int_0^1 x^{33}(1-x)^{17}dx \\ &= B(18,34) = B(34,18) \\ &= \frac{\Gamma(18)\Gamma(34)}{\Gamma(52)} \\ &= \frac{17!\cdot 33!}{51!} \\ &= \frac{1}{51\cdot\begin{pmatrix}50\\17\end{pmatrix}} \\ &= \frac{1}{51\cdot\begin{pmatrix}50\\33\end{pmatrix}}\end{aligned}$$

3.

$$\begin{aligned}\int_0^1 x^{\frac{1}{3}}(1-x)^{\frac{7}{5}}\,dx &= B\left(\frac{4}{3},\frac{12}{5}\right)\\ &= \frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{12}{5}\right)}{\Gamma\left(\frac{4}{3}+\frac{12}{5}\right)}\\ &= \frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{12}{5}\right)}{\Gamma\left(\frac{56}{15}\right)}.\end{aligned}$$

Now this can be evaluated approximately by using tables of the Gamma function or computer means.

4. By $(B,8)$, we readily get

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\frac{\pi}{\sin\left(\frac{1}{4}\pi\right)}=\pi\sqrt{2}$$

So,

$$\Gamma\left(\frac{3}{4}\right)=\frac{\pi\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)},\quad \text{or}\quad \Gamma\left(\frac{1}{4}\right)=\frac{\pi\sqrt{2}}{\Gamma\left(\frac{3}{4}\right)}$$

5. We use $(B,5),(B,7)$ and $(B,8)$ to find

$$
\begin{aligned}
\int_0^\infty \frac{\sqrt[4]{x}}{(1+x)^2} dx &= B\left(\frac{3}{4}, \frac{5}{4}\right) \\
&= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{5}{4}+\frac{3}{4}\right)} \\
&= \frac{\Gamma\left(1+\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(2)} \\
&= \frac{1}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{1} \\
&= \frac{1}{4} \cdot \Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) \\
&= \frac{1}{4} \cdot \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} \\
&= \frac{1}{4} \cdot \frac{\pi}{\frac{\sqrt{2}}{2}} \\
&= \frac{\pi\sqrt{2}}{4}.
\end{aligned}
$$

6. We use $(B, 6), (B, 7)$ and $(B, 8)$ to find

$$\begin{aligned}
\int_0^{\pi/2} \sin^{\frac{5}{2}}(x)\cos^{\frac{3}{2}}(x)dx &= \frac{1}{2}\cdot B\left(\frac{7}{4},\frac{5}{4}\right) \\
&= \frac{1}{2}\cdot\frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma(3)} \\
&= \frac{1}{2}\cdot\frac{\Gamma\left(1+\frac{3}{4}\right)\cdot\Gamma\left(1+\frac{1}{4}\right)}{2!} \\
&= \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) \\
&= \frac{3}{64}\cdot\frac{\pi}{\sin\left(\frac{\pi}{4}\right)} \\
&= \frac{3\pi\sqrt{2}}{64}.
\end{aligned}$$

7. By letting $q=\frac{1}{2}$ in $(B,6)$, then by $(B,7)$ we get that for $p>0$ constant

$$\int_0^{\pi/2}\sin^{2p-1}(\theta)d\theta = \frac{1}{2}B\left(p,\frac{1}{2}\right) = \frac{1}{2}\cdot\frac{\Gamma(p)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(p+\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(p)}{\Gamma\left(p+\frac{1}{2}\right)}.$$

From this, by the symmetry and the positivity of sine in $[0,\pi]$, we also obtain

$$\int_0^{\pi}\sin^{2p-1}(\theta)d\theta = B\left(p,\frac{1}{2}\right) = \frac{\Gamma(p)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(p+\frac{1}{2}\right)} = \frac{\sqrt{\pi}\Gamma(p)}{\Gamma\left(p+\frac{1}{2}\right)}.$$

Similarly for $q>0$ constant

$$\int_0^{\pi/2}\cos^{2q-1}(\theta)d\theta = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(q)}{\Gamma\left(q+\frac{1}{2}\right)}.$$

In this case, we cannot extend the integral over $[0,\pi]$ when the exponent is not an integer, since the cosine is negative in $(\frac{\pi}{2},\pi\,]$.

But, if the exponent is an even integer, then the integral over $[0, \pi]$ is twice the previous integral, and if the exponent is odd, the integral over $[0, \pi]$ is zero.

8. By $(B, 6), (B, 7)$ and exercise 1 of this section, for $m \geq 0$ and $n > 0$ integers, we find

$$\begin{aligned}
\int_0^{\pi/2} \sin^{2m}(x)\cos^{2n}(x)dx &= \frac{1}{2} \cdot B\left(m + \frac{1}{2}, n + \frac{1}{2}\right) \\
&= \frac{1}{2} \cdot \frac{\Gamma\left(m + \frac{1}{2}\right)\Gamma\left(n + \frac{1}{2}\right)}{\Gamma(m+n+1)} \\
&= \frac{\pi}{2^{2m+2n+1}} \cdot \frac{(2m)!(2n)!}{m!n!(m+n)!}.
\end{aligned}$$

The integral over $[0, \pi]$ is twice this integral.

9. In general, for $p > 0$ and $q > 0$ by $(B, 6)$ and $(B, 7)$, we have that:

$$\int_0^{\pi/2} \sin^{2p-1}(\theta)\cos^{2q-1}(\theta)d\theta = \frac{1}{2}B(p, q) = \frac{1}{2}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

10. $\forall\ C > 0, \forall D > 0, \forall p > 0, \forall q > 0$ constants, we have

$$\int_0^{\pi/2} \frac{\sin^{2p-1}(\theta)\cos^{2q-1}(\theta)}{\left[C\sin^2(\theta) + D\cos^2(\theta)\right]^{p+q}}d\theta = \frac{B(p, q)}{2C^p \cdot D^q}$$

This result is obtained by making the change of variables $u = \sin^2(\theta)$ and adjusting the integral obtained to $(B, 9)$.

EXERCISES

1. Prove directly, or by induction, or by using the byproduct of the duplication formula in $(B, 10)$, that:

   (a) For $n = 0, 1, 2, 3, \ldots$, we have

   $$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{2^{2n} n!} = \frac{(2n)!\sqrt{\pi}}{4^n n!}.$$

   This expression for $n = 1, 2, 3, \ldots$ simplifies to

   $$\Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n}\sqrt{\pi}.$$

   (b) Use the definition of $\Gamma(p)$ for $p < 0$ to prove that for $m = 1, 2, 3, \ldots$

   $$\Gamma\left(-m + \frac{1}{2}\right) = \frac{(-1)^m 2^m}{1 \cdot 3 \cdot 5 \cdots (2m-1)}\sqrt{\pi} = \frac{(-1)^m 2^{2m} m!}{(2m)!}\sqrt{\pi}.$$

2. For all $\alpha < 1$, prove the two equalities

   $$\int_0^{\pi} |\sec(x)|^{\alpha} dx = 2\int_0^{\pi/2} \sec^{\alpha}(x) dx = \sqrt{\pi} \cdot \frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{2-\alpha}{2}\right)},$$
   $$\int_0^{\pi} |\csc(x)|^{\alpha} dx = 2\int_0^{\pi/2} \csc^{\alpha}(x) dx = \sqrt{\pi} \cdot \frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{2-\alpha}{2}\right)}.$$

   Show that the formulae give the right answer (∞) even when $\alpha = 1$ and evaluate the results for $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{3}$.

3. Using the properties of the Beta and Gamma functions and $v^2 = \tan(x)$, prove

   $$\int_0^{\infty} \frac{dv}{1 + v^4} = \frac{\pi\sqrt{2}}{4}.$$

4. Use property $(\,\Gamma, 8)$ of the Gamma function and property $(B, 7)$ of the Beta function to prove the recursive properties, $(B, 3)$ and $(B, 4)$, of the Beta function.

5. Compute
$$\int_0^\infty \frac{e^{-t}}{t^{\frac{5}{4}}}\,dt \text{ and } \int_0^\infty \frac{e^{-8t}}{(5t)^{\frac{1}{4}}}\,dt.$$

6. Use $x = b\tan(\theta)$ to prove that if $b > 0$, then
$$\int_0^b \frac{dx}{\sqrt{b^4 - x^4}} = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{4b\sqrt{2\pi}}.$$

7. Use $x^2 = b^2\tan(\theta)$ to prove that if $b > 0$, then
$$\int_0^\infty \frac{dx}{\sqrt{b^4 + x^4}} = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{4b\sqrt{\pi}}.$$

Syllabus

B.Sc. Mathematics: SEMESTER 6

MTS6 B10 : REAL ANALYSIS

No. of Hours of Lectures/week: 5
No. of Credits: 4
100 Marks [Int:20+Ext:80]

Aims, Objectives and Outcomes

The course is built upon the foundation laid in Basic Analysis course of fifth semester. The course thoroughly exposes one to the rigour and methods of an analysis course. One has to understand definitions and theorems of text and study examples well to acquire skills in various problem solving techniques. The course will teach one how to combine different definitions, theorems and techniques to solve problems one has never seen before. One shall acquire ability to realise when and how to apply a particular theorem and how to avoid common errors and pitfalls. The course will prepare students to formulate and present the ideas of mathematics and to communicate them elegantly.

On successful completion of the course, students will be able to

- State the definition of continuous functions, formulate sequential criteria for continuity and prove or disprove continuity of functions using this criteria.
- Understand several deep and fundamental results of continuous functions on intervals such as boundedness theorem, maximum-minimum theorem, intermediate value theorem, preservation of interval theorem and so on.
- Realise the difference between continuity and uniform continuity and equivalence of these ideas for functions on closed and bounded interval.
- Understand the significance of uniform continuity in continuous extension theorem.
- Develop the notion of Riemann integrability of a function using the idea of tagged partitions and calculate the integral value of some simple functions using the definition.
- Understand a few basic and fundamental results of integration theory.
- Formulate Cauchy criteria for integrability and a few applications of it. In particular they learn to use Cauchy criteria in proving the non integrability of certain functions.
- Understand classes of functions that are always integrable
- Understand two forms of fundamental theorem of calculus and their significance in the practical problem of evaluation of an integral.
- Find a justification for 'change of variable formula' used in the practical problem of evaluation of an integral.

- Prove convergence and divergence of sequences of functions and series - Understand the difference between pointwise and uniform convergence of sequences and series of functions
- Answer a few questions related to interchange of limits.
- Learn and find out examples/counter examples to prove or disprove the validity of several mathematical statements that arise naturally in the process/context of learning.
- Understand the notion of improper integrals, their convergence, principal value and evaluation.
- Learn the properties of and relationship among two important improper integrals namely beta and gamma functions that frequently appear in mathematics, statistics, science and engineering.

Syllabus

Text(1)	Introduction to Real Analysis(4/e) : Robert G Bartle, Donald R Sherbert John Wiley & Sons(2011) ISBN 978-0-471-43331-6
Text(2)	Improper Riemann Integrals: Ioannis M. Roussos CRC Press by Taylor & Francis Group, LLC(2014) ISBN: 978-1-4665-8808-0 (eBook - PDF)

Module-I Text(1) (18 hrs)

5.1: Continuous Functions- definition, sequential criteria for continuity, discontinuity criteria, examples of continuous and discontinuous functions, Dirichlet and Thomae function

5.3: Continuous Functions on Intervals- Boundedness Theorem, The Maximum-Minimum Theorem, Location of Roots Theorem, Bolzano's Intermediate Value Theorem, Preservation of Intervals Theorem

5.4: Uniform Continuity- definition, illustration, Nonuniform Continuity Criteria, Uniform Continuity Theorem, Lipschitz Functions, Uniform Continuity of Lipschitz Functions, converse, The Continuous Extension Theorem, Approximation by step functions & piecewise linear functions, Weierstrass Approximation Theorem (only statement)

Module-II Text(1) (22 hrs)

7.1: Riemann Integral -Partitions and Tagged Partitions, Riemann sum, Riemann integrability, examples, Some Properties of the Integral, Boundedness Theorem

7.2: Riemann Integrable Functions-Cauchy Criterion, illustrations, The Squeeze Theorem, Classes of Riemann Integrable Functions, integrability of continuous and monotone functions, The Additivity Theorem

7.3: The Fundamental Theorem-The Fundamental Theorem (First Form), The Fundamental Theorem (Second Form), , Substitution Theorem, Lebesgue's lntegrability Criterion, Composition Theorem, The Product Theorem, Integration by Parts, Taylor's Theorem with the Remainder.

Module-III Text(1) (17 hrs)

8.1: Pointwise and Uniform Convergence-definition, illustrations, The Uniform Norm, Cauchy Criterion for Uniform Convergence

8.2: Interchange of Limits- examples leading to the idea, Interchange of Limit and Continuity, Interchange of Limit and Derivative [only statement of theorem 8.2.3 required; proof omitted] Interchange of Limit and Integral, Bounded convergence theorem(statement only) [8.2.6 Dini's theorem omitted]

9.4: Series of Functions - (A quick review of series of real numbers of section 3.7 without proof) definition, sequence of partial sum, convergence, absolute and uniform convergence, Tests for Uniform Convergence, Weierstrass M-Test (only upto and including 9.4.6)

Module-IV Text(2) (23 hrs)

Improper Riemann Integrals 1.1: Definitions and Examples

1.2: Cauchy Principal Value

1.3: Some Criteria of Existence

2.1: Calculus Techniques ['2.1.1 Applications' Omitted]

2.2: Integrals Dependent on Parameters- upto and including example 2.2.4

2.6: The Real Gamma and Beta Functions- upto and including Example 2.6.18

References:	
1	Charles G. Denlinger: Elements of Real Analysis Jones and Bartlett Publishers Sudbury, Massachusetts (2011) ISBN:0-7637-7947-4 [Indian edition: ISBN-9380853157]
2	David Alexander Brannan: A First Course in Mathematical Analysis Cambridge University Press, US(2006) ISBN: 9780521684248
3	John M. Howie: Real Analysis Springer Science & Business Media(2012)[Springer Undergraduate Mathematics Series] ISBN: 1447103416
4	James S. Howland: Basic Real Analysis Jones and Bartlett Publishers Sudbury, Massachusetts (2010) ISBN:0-7637-7318-2
5	Terence Tao: Analysis I & II (3/e) TRIM 37 & 38 Springer Science+Business Media Singapore 2016; Hindustan book agency(2015) ISBN 978-981-10-1789-6 (eBook) & ISBN 978-981-10-1804-6 (eBook)
6	Richard R Goldberg: Methods of Real Analysis Oxford and IBH Publishing Co.Pvt.Ltd. NewDelhi(1970)

7	Saminathan Ponnusamy: Foundations of Mathematical Analysis Birkhauser(2012) ISBN 978-0-8176-8291-0
8	William F Trench: Introduction to Real Analysis ISBN 0-13-045786-8
9	Ajith Kumar & S Kumaresan : A Basic Course in Real Analysis CRC Press, Taylor & Francis Group(2014) ISBN: 978-1-4822-1638-7(eBook - PDF)
10	Hugo D Junghenn : A Course in Real Analysis CRC Press, Taylor & Francis Group(2015) ISBN: 978-1-4822-1928-9 (eBook - PDF)