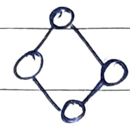
SETS

SET BUILDER NOTATION



MST
(tree, no cycle)
kruskal, prims

- enumerating items
  ↳ set may be too big

infinite

$\{x \mid x \text{ is an even integer}\}$

allows you to enumerate

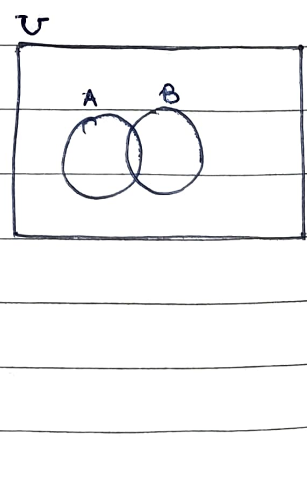
$\{a, e, i, o, u\}$

$\{1, 2, 3, \ldots\}$ ← natural numbers (unambiguous)

$\mathbb{Z}$ - set of integers.
$\mathbb{N}$ - set of natural numbers
$\mathbb{Q}$ - set of integers (rational number)
$\mathbb{R}$ - set of real numbers
$\mathbb{R}^+$ - set of +ve real numbers
$\mathbb{C}$ - set of complex numbers

ϕ - empty set / null set ← symbol for $\{\}$

U - Universe set



SETS AND LOGIC

equivalences of sets

ex- $\underbrace{A \cup B = B \cup A}_{\text{how?}}$ commutativity

OPERATIONS

$\bar{A} = U - A$ complement

$A \cup B = \{x \mid x \in A$ or $x \in B$ or both$\}$

$A \cap B = \{x \mid x \in A$ and $x \in B\}$
as well as

ex Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
using properties of logic.

let $x \in A \cup (B \cap C)$

$x \in A$ or $x \in B \cap C$

$x \in A \vee x \in B \cap C$

$x \in A \vee (x \in B \wedge x \in C)$

using distributivity of logical 'or' and 'and'

$(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$

RHS $= x \in A \cup B \wedge x \in A \cup C$

$= (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$

$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

de morgan's laws
equivalent for set

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

ex prove $A \cup (B_1 \cap B_2 \cap \dots B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \dots (A \cup B_n)$

proof by induction

base case $(n=2)$ $A \cup (B_1 \cap B_2) = (A \cup B_1) \cap (A \cup B_2)$

inductive assumption:

assume that the rule is true for K ← natural number

$A \cup (B_1 \cap B_2 \cap \dots B_k) = (A \cup B_1) \cap (A \cup B_2) \cap \dots (A \cup B_k)$.

RTP required to prove

this is true for $K+1$

$A \cup (B_1 \cap B_2 \cap \dots B_{k+1})$

distributivity

$= (A \cup (B_1 \cap B_2 \cap \dots B_k)) \cap (A \cup B_{k+1})$

from assumption

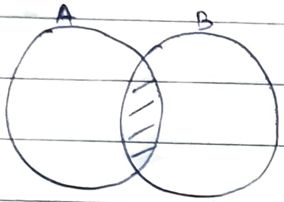
$= (A \cup B_1) \cap (A \cup B_2) \cap \dots (A \cup B_k) \cap (A \cup B_{k+1})$

therefore by induction it is true for any natural number n.

re iterated link between sets & logic.

PRINCIPLE OF INCLUSION EXCLUSION

counting principle



cardinality of set

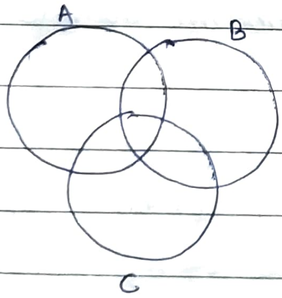
= no. of elements in the set

= $|A|$

$$|A \cup B| = \underbrace{|A| + |B|}_{\text{inclusion}} - \underbrace{|A \cap B|}_{\text{exclusion}}$$

trick:

count what you don't want to count.



trick:

count multiple times, take out the extra you counted.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$\therefore$ general

ex derangement:

how many ways to give back quiz so that no one gets back their own quiz.

43

ex 12 people, of which
10 smart people, 5 cs students
how many are both smart and cs students.

$|A \cup B| = 12$

$|A| + |B| - |A \cap B| = 12$

$10 + 5 - |A \cap B| = 12$

$|A \cap B| = 15 - 12 = \underline{\underline{3}}$

ex how many numbers in 1-1000 are divisible by
3, 5, or 7.

$|N_{d_3}| = |A|$, $|N_{d_5}| = |B|$, $|N_{d_7}| = |C|$

$$\left.\begin{aligned} |A| &= \left\lfloor \frac{1000}{3} \right\rfloor = 333 \\ |B| &= \left\lfloor \frac{1000}{5} \right\rfloor = 200 \\ |C| &= \left\lfloor \frac{1000}{7} \right\rfloor = 142 \end{aligned}\right\} 675$$

$|A \cap B| = \left\lfloor \frac{1000}{15} \right\rfloor = 66$ $\qquad$ $|B \cap C| = \left\lfloor \frac{1000}{35} \right\rfloor = 28$

$|A \cap C| = \left\lfloor \frac{1000}{21} \right\rfloor = 47$ $\qquad$ 141

$|A \cap B \cap C| = \left[\frac{1000}{105} \right] = 9$

$\therefore |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= 333 + 200 + 142 - 66 - 47 - 28 + 9$

$= \underline{\underline{543}}$

FUNCTION

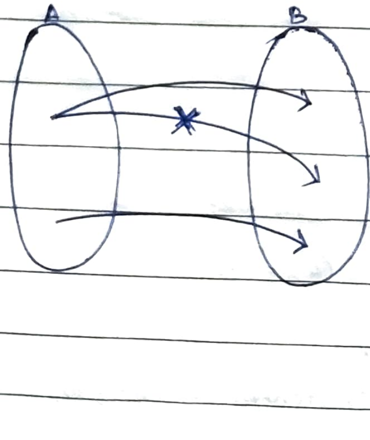
definition $f: A \to B$

consist of : a set A, a set B

and a subset $G(f)$ of $A \times B$

graph of function

with the property that each member of A is the first member of exactly one ordered pair belonging to $G(f)$



$G(f)$ = set of all ordered pairs

$G(f) = \{(a,b) \mid (a,b) \in A \times B\}$

$G(f) \subseteq A \times B$

A: domain of the function

B: codomain of the function.

if $(a,b) \in G(f)$

$f(a) = b$

pre-image, image

$G(f) \subseteq A \times B$

is called the graph of the function

(a,b)

↳ image of 'a' under function 'f'

set of all images called the "range" of f.

range $\subseteq$ codomain

$\text{range}(f) \subseteq B$

<u>dfn</u> let $f: A \to B$ be a function

f is <u>one-to-one</u> function (injective / injection) provided, whenever $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$ then $a_1 = a_2$.

<u>dfn</u> f is said to be <u>onto</u> or surjection, provided that each member of B is the image of atleast one member of A.

$\text{range}(f) = B$ (codomain)

$$\underbrace{\overset{\text{for all}}{\forall}\, b \in B}_{\text{universal quantifier}},\ \underbrace{\overset{\text{there exists}}{\exists}\, a \in A}_{\text{existential quantifier}}\ \overset{\text{such that}}{\ni}\ f(a) = b$$

predicate logic

quantifiers

relations of items

teacher of DME

student of DME

1^{st} order
2^{nd} order
most used

higher order

PROBLEM

translate to english equivalent.

PROLOG.

↳ uses predicate logic

statement about word

used as predicate

dfn a function that is both one-to-one (injection) and onto (surjection) is called a one-to-one correspondence (bijection).

dfn real-valued function

↳ codomain = $\mathbb{R}$

dfn complex-valued function

↳ codomain = $\mathbb{C}$

dfn integer-valued function

↳ codomain = $\mathbb{Q}$

ex let $f: \mathbb{R} \to \mathbb{R}$

be defined by role $f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x^2 - x & \text{if } x > 0 \end{cases}$

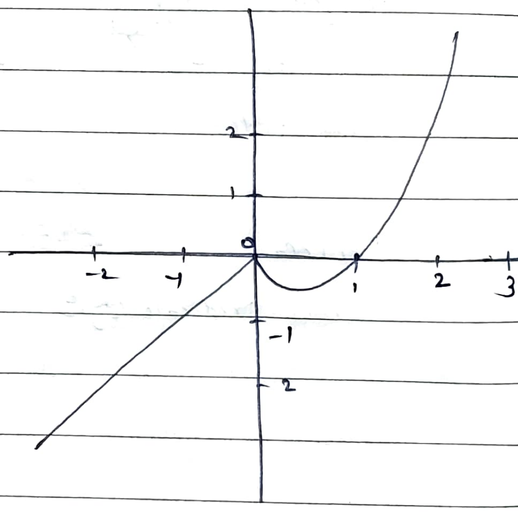
(i) is f one-to-one function?

(ii) is it onto?

(iii) find $f(S)$ when $S = [-1, 0]$

$S = \{-1, 1, 0\}$

$S = \mathbb{R}$



(i) $f(0) = 0$

$f(1) = 0$

however $0 \neq 1$

$\Rightarrow$ f is not one-to-one function

(Counter example)

(ii) RTP:

for every image there is atleast one pre-image

let $r \in \mathbb{R}$

if $r \leq 0$ $f(r) = r$

$\hookrightarrow x = r$ when $r \leq 0$

pere-image

if $r > 0$, $\exists\, x \in \mathbb{R} \ni f(x) = r$

$x^2 - x = r$

$x^2 - x - r = 0$

$$x = \frac{1 + \sqrt{1+4r}}{2}$$

pre-image for r, when $r > 0$

ex- $r = 2$ $\quad x = \frac{1 + \sqrt{9}}{2} = 2$

given any value from codomain, one can find a pre-image.

$\therefore$ it is onto function.

(iii) find $f(S)$ when $S = [-1, 0]$

$S = \{-1, 1, 0\}$

$S = \mathbb{R}$

$f([-1,0]) = [-1, 0]$

$f(\{-1, 1, 0\}) = \{-1, 0\}$ — don't repeat

$f(\mathbb{R}) = \mathbb{R}$ (since f is onto)

completely exhausts the codomain.